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Master Thesis

Generalised Geometry, Parallelisations and Consistent Truncations

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"[That's] what distinguishes the good theorists from the bad ones. The good ones always make an even number of sign errors, and the bad ones always make an odd number."

Anthony Zee, *Quantum Field Theory in a nutshell*

To Camilla, who will make her way.
To my land, to all men from the South, wherever they come from.

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Abstract

Faculty of Natural Sciences
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Generalised Geometry, Parallelisations and Consistent Truncations

by Oscar de Felice

In this thesis we develop and use the tools of generalised geometry. First of all we review construction and symmetries of these structures, defining the key elements of generalised geometry, including the notion of torsion-free generalised connections, and show how this geometry can be used to give a unified description of the supergravity fields, exhibiting an enlarged local symmetry group. The first part of the work will be ended showing how the equations of motion for the NSNS sector of Type II Supergravity theories can be elegantly expressed in the framework of generalised geometry in the same form of the Einstein's equations of motion for gravity in ordinary geometry. In the second part we investigate the notion of "Leibnitz generalised parallelisations", the analogue of a local group manifold structure in generalised geometry, aiming to characterise completely such class of manifolds, which play a central role in the study of consistent truncations of supergravity. Original results of this works are examples of Leibnitz parallelisms for the manifolds $S^2 \times S^1$, $S^3 \times S^3$ and homogeneous spaces AdS_3, dS_3 and H^3 which, according to a conjecture recently formulated, should provide consistent truncations of supergravity theories. As conclusion of this part, we formulate and analise a proposition by Gibbson, Pope et al. in the context of generalised geometry, showing how it can be better investigated with this new instruments.

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Contents

Introduction

Nowadays, in the most agreed picture of the nature, physicists believe in the existence of four fundamental forces, gravitational, electromagnetic, strong nuclear and weak nuclear. The main open problem in theoretical physics in the last half century is certainly the description of the four fundamental forces in a single unified framework. For the last three interactions, we have a theory that describes them in quite a satisfactory way, even if several points remain obscure. This theory makes use of the *Quantum Field Theory* and is commonly known as the *Standard Model.* It is, as far as we know, the most complete, experimentally verifiable theory of fundamental interactions. Currently, the Standard Model is the agreed picture of fundamental physics among the physicists, also thanks to the important results from CERN and from many other laboratories. It has the important property of being *renormalisable.* In fact, some infinite quantities appears in calculating, for instance, correlators with perturbative methods. This problem afflicted many of the most brilliant minds of the last century, since correlation functions are related to observable quantities, that, of course, cannot be divergent. *Renormalisation* is the procedure that removes these divergencies: although ultraviolet divergencies exist they can be enclosed into a finite set of parameters, called sometimes regularisation parameters, such that physical quantities are finite, then the renormalisation consists into the redefinition of the constants of the theory (mass, charge, etc.) such that the regularisation parameters we introduced are removed.
Obviously, world cannot be so simple. There are several conditions a theory must satisfy in order to be renormalisable, and Einstein's general relativity, the currently accepted theory of gravity, is not renormalisable.

Several extensions of Standard model have been proposed in order to incorporate gravity into a quantum field description, but none of them can be confirmed by experiments yet, however a larger and larger number of physicists believe that String Theory – or better, SuperString Theory – can provide a good unified explanation of all the four fundamental interactions.
The fundamental objects in string theory are not the point particle of the standard

model, but finite dimensional entities (strings or branes) and their vibrational modes are what we know for fields. This simple idea has a lot of very deep and still not completely understood and discovered implications, but the theory seems to be able to unify all forces of nature.
String theory was born in the sixties in order to find an explanation to some complex phenomena related to hadrons behaviour, it has been reconsidered in the second half of the seventies since Schwarz and Scherk realised it could explain some properties of the particle supposed to mediate the gravitational force, the graviton.
At the beginning the field content of string theory was just bosonic and, for reason of consistency, the number of spacetime dimensions had to be fixed to 26. However, this theory has several aspects that make it unsatisfactory. Above all, it admits tachyons, that is singularities in the spacetime, contrasting special relativity, and it does not consider the existence of fermionic particles. To solve these issues and to introduce fermions in string theory, we need supersymmetry for consistency. This gives origin to superstring theory, in which spacetime is 10 dimensional. Introduction of supersymmetry, and the development of superstring theories is known as *first string revolution*.
Thus, modern string theory is based on supersymmetry, a deep symmetry between bosons and fermions, whose one of the most famous consequences is the (expected) existence of a *superpartner* for each existing particle in nature, following the opposite statistics. This hypotesis solves elegantly and admirably the problem of renormalisation, since for each divergent loop diagram there exists a corresponding diagram that contributes in the same way, but with opposite sign, because of the statistics, to the total correlator. Thus, the divergent diagrams cancel each other as in figure 1.

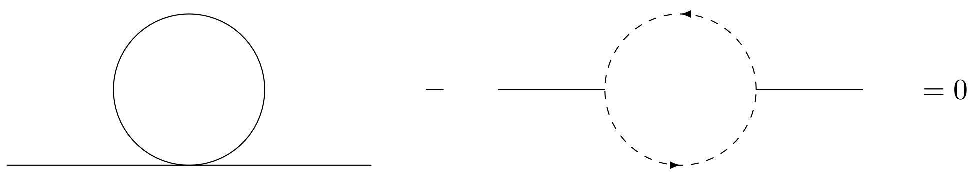

Figure 1: Loop diagram cancellation.

Another strong point of string theory is the possibility of avoiding short distances singularities. In fact, substituting point particles with extended object, we get a worldsheet that is a smooth manifold and the interaction vertices are given by diagrams as in figure 2. The price we pay is the loss of sharpness in the spacetime localisation of the interaction, nonetheless we have not to cope with singularities due to a short-distance scale of interaction, since the scale of the string ℓ_s defines naturally a *cut-off* scale $\Lambda \sim \ell_s^{-1}$.

The fact that the consistency of the theory requires the spacetime to be 10 dimensional can be overcome by a procedure called *compactification*. It consists in curling extra-dimensions on compact manifold with characteristic volume such that they are

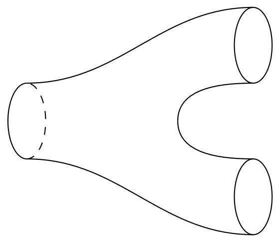

Figure 2: Feynmann diagram of a string interaction vertex.

not perceptible at the energy-length scales examinable by measures.
This leads us to the concepts of *dimensional reduction* and *consistent truncations*, better analysed in the main text.
The *dimensional reduction* is a procedure that allows us to define a field theory in a spacetime of dimension $d \leq 10$. Our aim is to produce a theory in 4 dimensions, since this is the dimension of the physical world we experience everyday. The dimensional reduction consists in taking the limit of the characteristic volume of the compactification space and make it smaller than the scale Λ defined above. This produces, given a massless field in the higher dimensional space, a tower of massive states in lower dimensional theory. A *consistent truncation* is a finite subset of modes, where the omitted heavy modes are not sourced by the light ones. In other words, we have found two sets of modes, and one of them has a dynamics which is independent from the other, this gives origin to a decoupled consistent theory constructed by one set of modes only. The last statement is the reason of our interest in consistent truncations, in fact, any solution to the equations of motion expressed by only consistent truncated modes in the lower dimensional theory remains a solution when uplifted in the higher dimensional space.
Moreover, it is worth to notice that the theory encoding both string theory and supersymmetry is not unique, there are five different superstring theories: type I, type IIA, type IIB, $SO(32)$ heterotic and $E_8 \times E_8$ heterotic.
In mid 1990s the concept of dualities arose in string theory, giving start to a process called *second string revolution*, and this allowed to discover that superstring theories are actually different limits of a more fundamental theory, living in an 11-dimensional spacetime, called M-theory. The five superstring theories are, in fact, related by dualities.

In this work we are going to analyse principally the bosonic part of the spectrum of the type IIA and IIB theories, made, for both, by the fields $\{g_{\mu\nu}, B_{\mu\nu}, \phi\}$. The remaining part of the spectrum is what makes the IIA and IIB thoeries different and is known as Ramond-Ramond sector. To be more precise, we are going to take into account the low energy limit of these theories, the supergravity type IIA and IIB.
As the name may suggest, *supergravity* is a theory encoding both Einstein gravity and

supersymmetry (making this a pointwise symmetry). It was discovered independently from string theory in mid seventies, and only at a later stage physicists realised it was a limit at low energy scales of the latter.

We are going to use a particular mathematical environment, the so-called *Generalised Geometry*. It was firstly introduced by Hitchin and Gualtieri in an effort to study invariant functionals on differential forms in differential geometry, and then Hull, Waldram, et al. realised that it provides an extremely powerful and elegant tool to reformulate and better understand supergravity and string theory, stressing certain important symmetries of the theory. In particular, we are going to focus on the type II supergravity symmetries: the invariance under diffeomorphisms and the gauge invariance, showing how is possible encapsulating them into a single geometric object.

The principal idea of generalised geometry is, roughly speaking, to promote the tangent bundle TM to a generalised version of it built by the direct sum $TM \oplus T^*M$, and then develop a differential geometry on the generalised structure, proceding by analogies. In this work we will go through the construction of generalised objects analogous to the ones we are used to in differential geometry, like the metric, the Lie derivative, the connection, etc. After that, we are going to analyse consistent truncations and their relation to generalised geometry.

An important property of consistent truncations is the fact that their existence is only related to the choice of the compactification manifold. Although this nice property, examples of consistent truncations are not so copious, and still not well understood. Thus, we are going to study a conjecture that links a special class of generalised parallelisable manifolds in generalised geometry to consistent truncations in string theory.

Finally, we will study a conjecture about gauged theories coming from compactification of higher dimensional theories, due to Gibbson, Pope et al. and we will show how in the context of generalised geometry it takes a natural and more elegant form.

There are several reasons to study generalised geometry further than these we indicated above, for example, it has been proved that generalised geometry is locally equivalent to *double field theory*, a particular kind of field theory, developed by Hull and Zwiebach, that lives in a doubled configuration space with coordinates $(x, \tilde{x})$ and so $2d$-dimensional. The remarkable fact is that both theories share the same (generalised) tangent bundle with an $O(d, d)$ group structure.

The group $O(d, d)$ is also the group related to T-duality, and this gives a fashionable hint to relate generalised geometry with the description of dualities. However, we have to stress the fact that actually the $O(d, d)$ groups appearing in the two contexts are different. In generalised geometry it is the structure group acting on the fibers of the tangent bundle, on the other hand, in the T-duality case the group encodes the symmetry between string and background configurations, when a string theory is compactified, for example on a torus T, and so the string has some configurations where it can be wrapped around the torus.

To summarise, this work has the aim of reviewing the principal aspects of generalised geometry, and, taking advantage of this review, investigating the concepts of Leibnitz parallelism and consistent truncations.

Chapter 1

Mathematical Environment: Introduction to Generalised Geometry

In this chapter we are going to introduce the most important tools for this thesis. First of all, we are going to define the generalised tangent bundle [1, 2], on which we will construct the analogues of the objects we use in ordinary differential geometry.
Following [2], we will show that a metric emerges naturally on these structures, defining a principal bundle, so we will study the behaviour of the frame bundle under the action of the structure group.
At conclusion of the chapter there is a section about the analogous of Riemannian metric in the context of generalised geometry. There we will present two different constructions of the generalised metric.

1.1 Generalised Tangent Bundle

Let us introduce the concept of Generalised Tangent Bundle. The basic idea of generalised geometry is to substitute the usual tangent bundle TM with the direct sum of TM and its dual space T^*M. This has several deep implications as regards geometric structures, and our aim is to analyse them throughout this project.
Formally, the direct sum $TM \oplus T^*M$ is not wat we call generalised tangent bundle, but we are not worried about this at this stage. We will come back later on this point.

Let M be a real differentiable manifold of dimension d, let TM be its tangent bundle, then consider the direct sum of the tangent and cotangent bundles $TM \oplus T^*M$.
As in [3], this kind of construction gives rise to a bundle. So a section of $TM \oplus T^*M$ can be written as the formal sum

$$V = v + \lambda.$$

Here, we indicated with v and λ sections of TM and T^*M respectively.

1.1.1 Linear Structure: the natural metric

Since TM and T^*M are vector space, related by a duality relation, this leads to a natural symmetric bilinear form [4] on $TM \oplus T^*M$

$$\langle V, W \rangle = \langle v + \lambda, w + \mu \rangle := \frac{1}{2}\left(\lambda(w) + \mu(v)\right). \tag{1.1}$$

Where, we denoted $\lambda(w)$ as the contraction $i_w\lambda$, and $i_w : \Lambda^r T^*M \to \Lambda^{r-1}T^*M$. Given an r-form with components $\omega_{\mu_1\ldots\mu_r}$ in a basis, the contraction $i_x\omega$ is an $(r-1)$-form which has components $x^\alpha\omega_{\alpha\mu_1\ldots\mu_{r-1}}$ in the same basis. The factor $1/2$ is chosen for convention and has no geometrical meaning.
This provides a non-degenerate inner product with signature (d, d) and therefore leads to define a natural metric tensor as

$$\eta_{AB} = \frac{1}{2}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\;, \tag{1.2}$$

that after a diagonalisation process becomes

$$\tilde{\eta}_{AB} = \frac{1}{2}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\;,$$

in which signature becomes explicit.
Defining a metric on a space is equivalent to set a group, in fact, metric can be seen as the invariant tensor of a particular group of matrices. In our case, the metric defined above characterises the group $O(TM \oplus T^*M) \cong O(d, d)$.

The $O(d, d)$ structure naturally arises from the linear structure of $TM \oplus T^*M$, in fact, this generalised bundle is the analog of the tangent one, and we can consider the group $O(d, d)$ as the structure group that acts on the generalised tangent bundle, in analogy with $GL(d, \mathbb{R})$ acting on the fibres of TM in the ordinary geometry case.

This leads to the concept of *tensors*, which in generalised geometry are representations of the $O(d,d)$ group.
Consider a frame (see later 1.3.1) $\{\hat{e}^a\}$ on TM and its dual $\{e_a\}$ on T^*M and define a frame on $TM \oplus T^*M$ as follows

$$\hat{E}_A = \begin{cases} \hat{e}^a & A = a \\ e_a & A = a + d \end{cases}$$

this lets us to express a generalised section of $TM \oplus T^*M$ as $V = V^A \hat{E}_A$. The metric can be used to raise and lower indeces. In fact contracting a generalised vector with the metric we get a generalised one-form $\eta_{AB} V^A = V_B$.
This provides an isomorphic map between $TM \oplus T^*M$ and $(TM \oplus T^*M)^*$, the "dual generalised tangent bundle". However, since

$$(TM \oplus T^*M)^* \cong TM \oplus T^*M \ ,$$

we can think V_A also as a generalised vector and we can write a generalised tensor in the following way

$$T^{A_1 \ldots A_r}_{B_1 \ldots B_s} \cong T^{A_1 \ldots A_r B_1 \ldots B_s} \in (TM \oplus T^*M)^{\otimes r+s} \ ,$$

where indeces run from 1 to $2d$. Thus, the last provide a representation of the $O(d,d)$ group, not necessarily irreducible.

Note that in addition to the product (1.1) we have pointed out, the generalised tangent bundle has a further structure: it is an orientable manifold too [4]. In fact, we can consider the highest antisymmetric tensor product of the generalised bundle

$$\Lambda^{2d} \left(TM \oplus T^*M\right).$$

It can be decomposed, in terms of antisymmetric tensor products of TM and T^*M only (since it is the maximal power) as

$$\Lambda^d TM \otimes \Lambda^d T^*M,$$

so we can argue that there is a natural pairing between two sections of $\Lambda^d TM$ and $\Lambda^d T^*M$ provided by the determinant map[1] [5].

[1] Actually, there is a natural pairing between $\Lambda^r TM$ and $\Lambda^r T^*M$ for any $1 \leqslant r \leqslant d$.

$$
\begin{aligned}
(\cdot,\cdot) : \Lambda^d TM \times \Lambda^d T^*M &\longrightarrow \mathbb{R} \\
v^*, w &\longmapsto \det(v_i^*(w_j))
\end{aligned}
$$

Thus we have an identification between $\Lambda^{2d}(TM \oplus T^*M)$ and $\mathbb{R}$; moreover $1 \in \mathbb{R}$ (or any other non-zero number) defines an orientation on $TM \oplus T^*M$.

The subgroup of $O(d,d)$ that preserves the invariant form η and the orientation is $SO(d,d)$, as we expected from the analogy we stated before between structure group $GL(d,\mathbb{R})$ for the conventional tangent bundle and $O(d,d)$. In fact, in the ordinary geometry case, the transformations that preserve orientation form the subgroup $SL(d,\mathbb{R}) \subset GL(d,\mathbb{R})$. Having an orientable space is equivalent to the existence of a top form nowhere vanishing, this allows us to define integration of differential form on the whole manifold, therefore they are useful objects to work with. Furthermore introducing spinors in this context, as in [6], will fix an orientation, suggesting that treating the subgroup $SO(d,d)$ is the right path to follow.

Before going on in our analysis, it is useful to make a remark about the nature of the structure we have exposed above.

Remark 1.1. Everything we stated until now for $TM \oplus T^*M$ descends from the duality between the two bundles and not from any other structures, thus all the properties are true also for a generalised space like $V \oplus V^*$ where V is a generic vector (real) space.

We have seen how the $O(d,d)$ structure always arises from the construction $TM \oplus T^*M$, now we restrict to the subgroup $SO(d,d)$, and firstly to its algebra, to study its decomposition and its action on the generalised bundle.

The Lie Algebra of the $SO(TM \oplus T^*M) \cong SO(d,d)$ is given by

$$
\mathfrak{so}(TM \oplus T^*M) = \{T \mid \langle TV, W\rangle + \langle V, TW\rangle = 0\}\,, \tag{1.3}
$$

i.e. generators are antisymmetric. This algebra decomposes [7] in

$$
End(TM) \oplus \Lambda^2 TM \oplus \Lambda^2 T^*M
$$

or, equivalently, a generic element $T \in \mathfrak{so}(TM \oplus T^*M)$ can be written as

$$
T = \begin{pmatrix} A & \beta \\ B & -A^T \end{pmatrix}\,, \tag{1.4}
$$

where $A \in End(TM)$, $B \in \Lambda^2 T^*M$, $\beta \in \Lambda^2 TM$, and hence

$$
\begin{aligned}
A : TM &\longrightarrow TM \\
B : TM &\longrightarrow T^*M \\
\beta : T^*M &\longrightarrow TM\,.
\end{aligned}
$$

We will be more interested in studying B transformations, as we will see, We can think to B as a 2-form acting on a section x of TM to give a 1-form as $B(x) := i_x B$. We will discuss this later and more in depth in section 1.2. We first need to put some differential structures on the generalised bundle, this will be the main goal of the next section.

1.1.2 Differential Structure: Dorfman derivative and Courant bracket

In the next, we are looking for an object that generalises the Lie derivative action in the conventional sense to generalised sections, these because we are moved by the aim of finding generalised version of objects we know in differential geometry, and developing them by analogy.

Given two sections of $TM \oplus T^*M$, as for example $V = v + \lambda$ and $W = w + \mu$, where $v, w \in \Gamma(TM)$ and $\lambda, \mu \in \Gamma(T^*M)$, we define the following object [1, 8]

$$L_V W := \mathcal{L}_v w + \mathcal{L}_v \mu - i_w d\lambda \tag{1.5}$$

Called *Dorfman derivative* [9].

This seems to play the same role as the Lie derivative in the ordinary geometry, in fact, it can be verified that it satisfies the Leibinitz rule for the product of two sections. The Leibinitz rule for Dorfman derivative can be arranged such that it resembles the Jacobi identity,

$$L_V(L_W U) = L_{L_V W} U + L_W(L_V U) \ .$$

However, at a more deep look, it appears that it is not antisymmetric, and therefore it does neither satisfy any *Jacobi identity*.

To solve this we can antisymmetrise the Dorfman derivative, defining a new object: the *Courant bracket*[10, 11].

$$[\![V, W]\!] := \frac{1}{2}\left(L_V W - L_W V\right). \tag{1.6}$$

This becomes the analogous of Lie bracket in the generalised geometry environment.

From this we can get also a deeper understanding of Lie structures. Lie derivative and Lie bracket are different objects: they coincide in ordinary geometry, but not in this generalised construction.

At this stage, we can point out a property that will be very useful in what follows, and that can be proved by a simple application of the definitions (1.5) and (1.6). The difference between the Courant bracket and the Dorfman derivative is the exterior derivative of the inner product, *i.e.*

$$[\![V, W]\!] = L_V W - \mathrm{d}\langle V, W\rangle \ . \tag{1.7}$$

Following our analogy, as Lie bracket is related to a Lie algebra, Courant bracket is connected to a structure called *Courant algebroid* [10, 12]. This kind of structures has been widely studied in mathematics, not only in the context of generalised geometry (see for example [13]) so in this work we will not analyse it extensively.
Although this, we can still give a vague idea of this interesting structure properties. We can use the definition **1.18** of [12] to introduce the *Courant algebroid*.[2]

Definition 1.2 (Courant algebroid)**.** A *Courant algebroid* over a manifold M is a vector bundle $\mathcal{E} \longrightarrow M$ endowed with an antisymmetric bracket $[\![\cdot,\cdot]\!]$, a non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$, and a projection map $\pi : \mathcal{E} \longrightarrow TM$ (often called **anchor**)[3], which satisfies the following conditions:

c1 $\pi\left([\![V,W]\!]\right) = [\pi(V), \pi(W)]$;

c2 $\mathrm{Jac}(V,W,Z) := [\![[\![V,W]\!],Z]\!] + \textit{circ. perm.} = \frac{1}{3}\mathrm{d}\left(\langle[\![V,W]\!],Z\rangle + \textit{circ. perm.}\right)$;

c3 $[\![V,fW]\!] = f[\![V,W]\!] + (\pi(V)[f])W - \langle V,W\rangle \mathrm{d}f, \qquad \forall f \in \mathcal{F}(M)$;

c4 $\pi \circ \mathrm{d} = 0$;

c5 $\pi(Z)\big[[\![V,W]\!]\big] = \langle adj(Z)[V],W\rangle + \langle V, adj(Z)[W]\rangle$.

Here, we have considered

$$adj(V)[W] := [\![V,W]\!] + \mathrm{d}\langle V,W\rangle$$

as the adjoint action of V on W, as suggested by the symbol. Moreover we denoted symbolically the further cyclic permutation in **c2** as *circ. perm.*

One of the main efforts of chapter 3 will be to study some cases when the frames from which we construct the Courant Algebroid actually define a Lie sub-algebra. This seems to have deep implications related to consistent truncations in string theory and supergravity, still not completely understood. The principal aim of this work is to investigate these implications, as we will see in what follows.
At this stage, what we can do is to verify that the Courant bracket we defined in (1.6) and our generalised tangent bundle $TM \oplus T^*M$ satisfy properties **c1**-**c5**, or equivalently, they define effectively a Courant algebroid.

Proposition 1.3. *The generalised tangent bundle $TM \oplus T^*M$ endowed with the $O(d,d)$ metric product* (1.1) *and with the bracket* (1.6) *defines a Courant algebroid.*

[2] A brief historical review of the subject can be found in the nice paper [14].
[3] This map must not be confused with the familiar bundle projection $E \xrightarrow{\pi} M$.

Proof (sketch). Here we only give a partial proof of the fact that the bracket (1.6) satisfies all the properties to define a Courant algebroid. A more complete proof can be found in [15] and in [2].

For our purposes it is useful to give a more explicit form of the Courant bracket we defined previously as

$$[\![V, W]\!] = [v, w] + \mathcal{L}_v \mu - \mathcal{L}_w \lambda + \frac{1}{2} \mathrm{d} \left(i_w \lambda - i_v \mu \right) . \tag{1.8}$$

Where, as above, $V = v + \lambda$, $W = w + \mu$.

c1

We can apply the anchor map π to the explicit form of the bracket given above, noting that the *vectorial* component (*i.e.* an element of TM) is made just by the usual Lie bracket.

$$\pi \big([\![V, W]\!] \big) = \pi \Big\{ [v, w] + \mathcal{L}_v \mu - \mathcal{L}_w \lambda + \frac{1}{2} \mathrm{d} \left(i_w \lambda - i_v \mu \right) \Big\} = [v, w] \, .$$

Recalling that $\pi(V) = v$ and $\pi(W) = w$, we have shown the property is satisfied.

c2

This property can be proved by direct calculation. The proof is quite long and not very instructive, so we omit it (see [16] for details).

c3

Given a function on the manifold $f : M \longrightarrow \mathbb{R}$ consider the following expression

$$[\![V, fW]\!] = [v, fw] + \mathcal{L}_v f \mu - \mathcal{L}_{fw} \lambda + \frac{1}{2} \mathrm{d} \left(i_{fw} \lambda - i_v f \mu \right) \ .$$

As usual, we use the known properties holding in the conventional geometry for the Lie derivative and for the inner product contraction,

$$\begin{aligned}
[\![V, fW]\!] &= [v, fw] + \mathcal{L}_v f \mu - \mathcal{L}_{fw} \lambda + \frac{1}{2} \mathrm{d} \left(i_{fw} \lambda - i_v f \mu \right) = \\
&= f [v, w] + v[f] \, w + v[f] \, \mu + f \mathcal{L}_v \mu - f \mathcal{L}_w \lambda - \mathrm{d}f \ i_w \lambda + \\
&+ \frac{1}{2} \mathrm{d}f \ i_w \lambda + \frac{1}{2} f \ \mathrm{d} i_w \lambda - \frac{1}{2} \mathrm{d}f \ i_v \mu - \frac{1}{2} f \ \mathrm{d} i_v \mu = \\
&= f [v, w] + v[f] \left(w + \mu \right) - \frac{1}{2} \mathrm{d}f \left(i_w \lambda + i_v \mu \right) + f \mathcal{L}_v \mu - f \mathcal{L}_w \lambda + \\
&+ \frac{1}{2} f \ \mathrm{d} \left(i_w \lambda - i_v \mu \right) .
\end{aligned}$$

By rearranging the last expression and writing $v = \pi(V)$, we can recognise the following quantity

$$[\![V, fW]\!] = f[\![V, W]\!] + (\pi(V)[f])W - \langle V, W\rangle \mathrm{d}f \ ,$$

as wanted.

We want to point out what this property implies about the difference between a Courant algebroid and a Lie algebroid [2]. The Courant bracket differs to be a Lie bracket by exact exact terms. In fact, in the case of a Lie structure we should have the known expression $[x, fy] = f\,[x, y] + x[f]y$.

c4

This property states that the exterior derivative d acting on functions, mapping a general f into a one-form $\mathrm{d}f$, will give us a section of the generalised bundle, made just by the form part. We will use this property explicitly to define generalised frames in section 1.3.

$$\pi\big(\mathrm{d}f\big) = 0 \ .$$

c5

Also this property can be proved by direct calculation and we do not report it. Refer to [15].

□

1.1.3 A basis for $TM \oplus T^*M$

In this brief section we want to introduce a basis, induced from the choice of a basis on TM and its dual on T^*M. This will be useful in what follows to keep track of the relation between generalised quantities and the usual ones in ordinary differential geometry.

Given a coordinate basis of TM on a chart U_i of M with components x^μ, $\{\partial/\partial x^\mu\}$ - often denoted for simplicity as $\{\partial_\mu\}$ - and its dual basis $\{dx^\mu\}$ on T^*M we can define a basis on $TM \oplus T^*M$ (restricted to U_i) as follows

$$\partial_A = \begin{cases} \partial_\mu & A = \mu \\ 0 & A = \mu + d \end{cases} \tag{1.9}$$

This provides an expression for the generalised derivative operator too, moreover, by dint of this we can give expression for Courant bracket and Dorfman derivative in terms of coordinates. For Dorfman derivative we have

$$(L_V W)^A = V^B \partial_B W^A + \left(\partial^A V_B - \partial_B V^A\right) W^B \tag{1.10}$$

and analogously for Courant bracket

$$[\![V, W]\!]^A = V^B \partial_B W^A - W^B \partial_B V^A - \frac{1}{2}\left(V_B \partial^A W^B - W_B \partial^A V^B\right). \tag{1.11}$$

Where index contractions are made using the metric (1.2), and so the expressions are $O(d,d)$ covariant.

In the (1.10) we can recognise the action of the adjoint representation of $\mathfrak{o}(d,d)$ by the matrix $\mathcal{A}^A{}_B := \left(\partial^A V_B - \partial_B V^A\right)$, analogously to the case of the Lie derivative in ordinary geometry.

This observation allows us to extend the definition of Dorfman derivative to the case of a generalised tensor,

$$\begin{aligned} L_V T^{A_1 \dots A_r} = V^B \partial_B T^{A_1 \dots A_r} &- \left(\partial^{A_1} V_B - \partial_B V^{A_1}\right) T^{B A_2 \dots A_r} - \dots + \\ &- \left(\partial^{A_r} V_B - \partial_B V^{A_r}\right) T^{A_1 \dots A_{r-1} B} \ . \end{aligned}$$

where we have considered just an $(r,0)$-tensor field, since, as argued above, because of $(TM \oplus T^*M)^* \equiv TM \oplus T^*M$, any (p,q)-generalised tensor can be rewritten as a $(p+q,0)$-generalised tensor.

1.2 Symmetries

Here, we would analyse symmetries of the structure we have defined above. In particular we are interested in determining the group of transformations that preserve the Courant bracket $[\![\cdot,\cdot]\!]$ and the inner product $\langle\cdot,\cdot\rangle$.

We have seen how the relevant algebra to consider is the (1.3), the $\mathfrak{so}\,(TM \oplus T^*M)$ algebra. We have also seen which form a generic element of the algebra should take in (1.4). We rewrite it here for our convenience

$$T = \begin{pmatrix} A & \beta \\ B & -A^T \end{pmatrix} \ .$$

We are now interested in considering a particular subclass of transformations among the whole algebra, as anticipated in the previous discussion. Consider the only action of the 2-form B

$$T_B = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} \ ,$$

in order to get an $SO(d,d)$ element we can use the exponential map, symbolically denoted as e^B, and observe how the B-field acts on the fibre of $TM \oplus T^*M$, noting that $(T_B)^2 = 0$ we can write

$$e^B := e^{T_B} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ B & \mathbb{1} \end{pmatrix} . \tag{1.12}$$

Then the action of the B-field on an element V of $\Gamma\left(TM \oplus T^*M\right)$, called *$B$-shift* is

$$e^B V = \begin{pmatrix} \mathbb{1} & 0 \\ B & \mathbb{1} \end{pmatrix} \begin{pmatrix} v \\ \lambda \end{pmatrix} = v + \lambda + i_v B . \tag{1.13}$$

An important result of this construction is that we have encoded into the $SO(d,d)$ transformations the old $GL(d)$ group action in conventional geometry. In fact, take into account another subset in $\mathfrak{so}(d,d)$

$$T_A := \begin{pmatrix} A & 0 \\ 0 & -A^T \end{pmatrix} ,$$

as usual, by the exponential map we find

$$e^{T_A} = \begin{pmatrix} e^A & 0 \\ 0 & e^{A^{-T}} \end{pmatrix} ,$$

where we indicated $A^{-T} = \left(A^T\right)^{-1}$.
Noting that $A \in GL\left(d, \mathbb{R}\right)$, and since $End(TM) \equiv GL\left(d, \mathbb{R}\right)$, we argue that $e^A, e^{A^{-T}} \in GL\left(d, \mathbb{R}\right)$ and so we can write them as ordinary matrices M, M^{-T}, and recalling

$$v' = Mv \qquad \lambda' = M^{-T}\lambda . \tag{1.14}$$

We can combine these two transformation into a generalised $SO(d,d)$ one,

$$e^{T_A} \begin{pmatrix} v \\ \lambda \end{pmatrix} = \begin{pmatrix} Mv \\ M^{-T}\lambda \end{pmatrix} . \tag{1.15}$$

Thence, we have showed that the usual $GL(d, \mathbb{R})$ transormations are embedded into a bigger group $O(d,d)$. There are, of course, other transformations like (1.13), which has been analysed previously, or the so called *β-shift* generated by elements in the algebra (1.3) taking the form

$$T_\beta := \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} .$$

Its exponential transformation acts on a generalised vector as $e^{T_\beta}\left(v + \lambda\right) \mapsto v + \lambda + \beta(\lambda)$ where for $\beta(\lambda)$ we mean (in components) the contraction between the 1-form λ and the

$(2,0)$-tensor β.
However, this kind of transformations will not play any important role in our discussion and we will not analyse it further [15].

1.2.1 Invariance of the Courant algebroid structure

Equipped with these tools, we can examine in some detail which kind of transormations in the $\mathfrak{so}(d,d)$-algebra leaves not only the inner product invariant (the whole algebra), but also the Courant bracket unchanged, finding so the group of symmetries of the generalised space structure $(TM \oplus T^*M,\ \langle\cdot,\cdot\rangle,\ [\![\cdot,\cdot]\!])$.

We have stated that any transformation in the algebra $\mathfrak{so}(d,d)$ leaves the inner product invariant. Let's verify this in particular for a B-shift. Consider, as usual, two generalised vectors $V = v + \lambda$ and $W = w + \mu$,

$$\begin{aligned}\langle e^B V,\ e^B W\rangle &= \langle v + \lambda + i_v B,\ w + \mu + i_w B\rangle = \\ &= \frac{1}{2}\big(i_v\left(\mu + i_w B\right) + i_w\left(\lambda + i_v B\right)\big) = \\ &= \frac{1}{2}\big(i_v\mu + i_w\lambda + i_v i_w B + i_w i_v B\big) = \\ &= \langle V, W\rangle\ .\end{aligned}$$

where, to go from the third line to the last one, we have used the antisymmetry of the inner product $i_v i_w$ when contracted with a 2-form as the B field, *i.e.* $i_v i_w B = -i_w i_v B$.

As stated, the whole group $O(d,d)$ preserves the inner product, hence, in order to find the subgroup preserving the Courant structure, we have to look at the Courant bracket. We state here a useful identity for the Lie derivative of forms of which we will make extensive use throughout this section, the so-called *Cartan identity*,

$$\mathcal{L}_v\omega = \big(i_v \mathrm{d} + \mathrm{d} i_v\big)\omega\ ,$$

that also implies $[\mathcal{L}_v, i_w] = i_{[v,w]}$.
Thus, let's apply a B-shift to two sections of $TM \oplus T^*M$, take their Courant bracket

and substitute the form we found in (1.8)

$$
\begin{aligned}
[\![e^B V, e^B W]\!] = [\![v+\lambda+i_v B,\ w+\mu+i_w B]\!] = \\
&= [\![v+\lambda,\ w+\mu]\!] + \mathcal{L}_v i_w B - \mathcal{L}_w i_v B - \frac{1}{2}\mathrm{d}\left(i_v i_w B - i_w i_v B\right) = \\
&= [\![v+\lambda,\ w+\mu]\!] + \mathcal{L}_v i_w B - \mathcal{L}_w i_v B - \mathrm{d}\left(i_v i_w B\right) = \\
&= [\![v+\lambda,\ w+\mu]\!] + \mathcal{L}_v i_w B - i_w \mathrm{d}\left(i_v B\right) = \\
&= [\![v+\lambda,\ w+\mu]\!] + i_{[v,w]} B + i_w \mathcal{L}_v B - i_w \mathrm{d}\left(i_v B\right) = \\
&= [\![v+\lambda,\ w+\mu]\!] + i_{[v,w]} B + i_w i_v \mathrm{d}B = \\
&= e^B\left([\![V,\ W]\!]\right) + i_w i_v \mathrm{d}B\ .
\end{aligned}
$$

This clearly show us that the B-shift is a symmetry of the structure if and only if $\mathrm{d}B = 0$, *i.e.* B has to be a closed 2-form.

We will show in the next how conciliate this with the request of having a non-trivial globally defined 3-form $H = \mathrm{d}B$. For now, let's point out an important observation:

Remark 1.4. The group that preserves the Courant bracket and the inner product at the same time is not the group of diffeomorphism on the manifold $Diff\,(M)$, but a bigger group, containg also B-field transformations, this group is $Diff\,(M) \ltimes Z^2\,(M)$ [17]. Here with $Z^2\,(M)$ we denoted the set of closed 2-form fields on M. It can also be proved that this is the only group that preserves this structure [2].

We can note also that the two factor in the semi-direct product $Diff\,(M) \ltimes Z^2\,(M)$ encodes different kind of transformations. In fact, diffeomorphisms take a point in the manifold and map it into another one, they are transformation *on* the manifold, whereas the B-shift is a *local* transformation, in the sense that it acts on fibre elements leaving the point on the manifold fixed; it is *pointwise*.

Finally, we should define something analogous to the pushforward of diffeomorphism for the general tangent space, in fact, recall for the conventional Lie bracket the invariance under pushforward can be represented by a commutative diagram[3]

$$
\begin{tikzcd}
TM \arrow[r, "f^*"] \arrow[d, "\pi"] & TM \arrow[d, "\pi"] \\
M \arrow[r, "f"] & M
\end{tikzcd}
\tag{1.16}
$$

where $f : M \to M$ is a diffeomorphism and $f^* : T_pM \to T_{f(p)}M$ its pushforward on the tangent space to M in the point p. Since f is a diffeomorphism we can actually define f^* acting on a vector field, not only on a vector, so $f^* : TM \to TM$.

The diagram (1.16) actually says that the Lie bracket "commutes" with the pushfoward action, in the following sense: given two vector fields x, y in TM,

$$f^*\big([x,\ y]\big) = [f^*x,\ f^*y] \qquad \forall x, y$$

Analogously for the generalised case. Recall that we define the pullback of a form field as $f_* : T^*M \to T^*M$, thus now we define a map $F : TM \oplus T^*M \to TM \oplus T^*M$ as $F := f^* \oplus f_*$ such that the Courant bracket "commutes", in the same sense of the Lie derivative, with this F action

$$F\left(\llbracket V,\ W \rrbracket\right) = \llbracket F\left(V\right),\ F\left(W\right) \rrbracket\ ,$$

where $F\left(V\right) = f^*v + f_*\lambda$. Just for completeness we can draw the commutative diagram in the generalised case.

$$\begin{tikzcd} TM \oplus T^*M \arrow[r, "F"] \arrow[d, "\pi"] & TM \oplus T^*M \arrow[d, "\pi"] \\ M \arrow[r, "f"] & M \end{tikzcd}$$

1.2.2 Patching rules

The usual approach [18] when we are in the presence of a geometrical structure with a symmetry is to restrict the structure locally, for instance on each open set of an open covering, and patch it with itself on an overlap between charts by the symmetry. We are going to follow this approach and study the Courant algebroid $(TM \oplus T^*M, \langle\cdot,\cdot\rangle, \llbracket\cdot,\cdot\rrbracket)$ restrictions and patching properties over an open cover $\{U_i\}$ of the base manifold M.
Take into account the generalised restricted space $TM \oplus T^*M\big|_{U_i}$ (sometimes denoted as $TU_i \oplus T^*U_i$) endowed with the Courant bracket and inner product we defined above, consider in addition $TM \oplus T^*M\big|_{U_j}$, where $U_i \cap U_j \neq \emptyset$ and sew them together on the overlapping region by the action of an element of the symmetry group, in particular we are interested in considering the B-field [19].
We can think at the B-field transformation that glues the two patches together as the transition function of a new bundle that is locally isomorphic to $TM \oplus T^*M$. We will return to this perspective at the end of this section. For now, we require the coherence of our patching procedure by imposing [20] a so-called *cocycle condition* on our transition "functions" on a triple intersection, like $U_i \cap U_j \cap U_k$

$$\begin{pmatrix} \mathbb{1} & 0 \\ B_{(ij)} & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ B_{(jk)} & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ B_{(ki)} & \mathbb{1} \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}\ .$$

This reduces to

$$B_{(ij)} + B_{(jk)} + B_{(ki)} = 0 \qquad \text{on } U_i \cap U_j \cap U_k\ . \tag{1.17}$$

Therefore we can form a new bundle, what we call the **Generalised tangent bundle** E defined as follows.

Definition 1.5 (Generalised Tangent Bundle)**.** The *generalised tangent bundle* E is defined as an extension of the tangent bundle TM by its cotangent bundle T^*M, given by the short exact sequence

$$0 \longrightarrow T^*M \xrightarrow{\iota} E \xrightarrow{\pi} TM \longrightarrow 0 \ ,$$

where π is the anchor map we introduced in the definition 1.2 and by ι we mean the trivial inclusion, i.e. $\iota(\lambda) = 0 + \lambda$, where 0 is the zero vector field in TM.
Given a point $p \in M$ we will denote the fibre at p as E_p, often called *generalised tangent space*.

There exists an important result in category theory, related to short exact sequences as the one written above, the *splitting lemma* [21]. This result is of crucial importance also for the generalised geometry since it allows us to prove the isomorphism between E and $TM \oplus T^*M$ (not only locally, as we expect from our construction). In order to formulate it properly we should need a lot of definitions and results that are over and above this project, so we remand to the literature for a complete rigorous treatment of the subject, for example [22, 23].
One important aspect to remark is that E has a (d,d) metric since each piece $TM \oplus T^*M\big|_{U_i}$ has the metric (1.2) and we patched charts by transformations in $SO(d,d)$, that do not affect the metric. It also have a Courant bracket on intersections since the forms $B_{(ij)}$ we used in the patching are closed, thus they preserve the bracket. To be rigorous, we should also check that the anchor map is actually the same we defined for $TM \oplus T^*M$, this is true indeed, since the B-shift does not affect the vector part of a generalised section, which means that $\pi\left(e^B V\right) = \pi\left(V\right) = v$.
Consider now the transition closed form $B_{(ij)}$ and find two forms $B_{(i)}$ and $B_{(j)}$ (not necessarily closed) defined respectively on U_i and U_j such that on the overlap region $U_i \cap U_j$ this provides a *local trivialisation* of $B_{(ij)}$, that is,

$$B_{(ij)} = B_{(i)} - B_{(j)} \ . \tag{1.18}$$

The key aspect of this *splitting* is that now we have a map (for each chart) that associate an element of TM to an element of E. We indicate this map as $B_{(i)}$ although the actual map is the matrix $e^{B_{(i)}}$, and dropping the chart index i, we write

$$\begin{aligned} B : TM &\longrightarrow E \\ v &\longmapsto v + i_v B \end{aligned}$$

Inserting this in the sequence defining the generalised tangent bundle,

$$\begin{tikzcd} 0 \arrow[r] & T^*M \arrow[r, "i"] & E \arrow[r, "\pi"] & TM \arrow[l, bend left, "B"] \arrow[r] & 0 \end{tikzcd} . \tag{1.19}$$

Hence, maps $B_{(i)}$ are only defined locally and we can patch them [24] on $U_i \cap U_j$ by

$$B_{(i)} = B_{(j)} - \mathrm{d}\Lambda_{(ij)} \ . \tag{1.20}$$

This patching rule means that we have chosen the closed 2-form $B_{(ij)}$ in (1.18) to be equal to the exact form $-\mathrm{d}\Lambda_{(ij)}$.
The cocycle condition (1.17) for the one forms $\Lambda_{(ij)}$, after integration, becomes

$$\Lambda_{(ij)} + \Lambda_{(jk)} + \Lambda_{(ki)} = \mathrm{d}\Lambda_{(ijk)} \qquad \text{on } U_i \cap U_j \cap U_k \ . \tag{1.21}$$

Where $\Lambda_{(ijk)}$ is a function.
The condition (1.21) is an analogous for one-forms transition "functions" of the patching requirements for a $U(1)$-bundle, and indeed to recall this, sometimes in literature (like in [24]) the one form $\mathrm{d}\Lambda_{(ijk)}$ is denoted, through a $U(1)$ element $g_{(ijk)} = e^{it}$, as $g_{(ijk)}\mathrm{d}g_{(ijk)}$.
This makes B a *connection structure on a gerbe* [20], that is to say the B is the two-form analogous of the $U(1)$ one-form connection A.
The choice of the patching through the one form $\Lambda_{(ij)}$ fixes the patching rules between sections of E over different charts of M. Take into account $v_{(i)} \in \Gamma(TU_i)$ and $\lambda_{(i)} \in \Gamma(T^*U_i)$ such that $V_{(i)}$ is a section of E over U_i, the patching of sections consistent with the connection structure given by B in (1.20) is

$$\begin{cases} v_{(i)} = v_{(j)} \\ \lambda_{(i)} = \lambda_{(j)} - i_{v_{(j)}}\mathrm{d}\Lambda_{(ij)} \end{cases} \tag{1.22}$$

on the overlap $U_i \cap U_j$. Note that we are not considering the usual $GL(d, \mathbb{R})$ transformation between different patches over the manifold.
The important aspect to notice is that while $v_{(i)}$ globally defines a vector field, $\lambda_{(i)}$ is not a proper globally defined one-form, since it does not patch in the required way for a one-form. In fact, E is isomorphic to $TM \oplus T^*M$ but this isomorphism is not natural, it is related to the choice of the splitting map B in the defining sequence 1.19. We can point out the relation between sections of $TM \oplus T^*M$ and elements of $\Gamma(E)$, sections of E, by the map B. In fact,

$$V = v + \lambda + i_v B = e^B \tilde{V} \ , \tag{1.23}$$

where we denoted as $\tilde{V} = v + \lambda$ a section of $TM \oplus T^*M$.

From the last expression we can see that the choice of a connection form B provides an isomorphism between $TM \oplus T^*M$ and E, as consequence of the splitting lemma [21, 25, 26]. We can see also that the isomorphism, as stated above, is crucially related to the choice of B. Moreover, this affects also the Courant bracket on E, which are said to be *twisted* by the choice of a trivialisation for the B-field, they are sligthly modified in [15]

$$[\![V, W]\!] =: e^B [\![\mathcal{V}, \mathcal{W}]\!]_H \ . \tag{1.24}$$

Where V, W are sections of E and $\mathcal{V}, \mathcal{W}$ are sections of $TM \oplus T^*M$. The three form $H := \mathrm{d}B$ is defined globally and is analogous to the field strength F of the A field in the case of a connection on a $U(1)$-bundle, as stated previously. It represents the field strength of the B-field and, in string theory, turns out to be quantised by a co-cyclicity condition on the quadruple intersection.
Hence, the *twisted bracket* (1.24) is defined globally on E thanks to the fact that on the intersection $U_i \cap U_j$ the $B_{(ij)}$ is a closed form and $H = 0$, otherwise the bracket would not patch properly [2].

1.3 Generalised Metric

In this section we look for a structure that plays an analogous role of the Riemannian metric in ondinary differential geometry. Therefore, we would like to put an "*extra structure*" on the generalised bundle, as the familiar metric structure on the manifold in differential geometry, and explore to what consequences the presence of this structure leads us.

1.3.1 Frames

As in ordinary geometry, we are interested in introducing an orthonormal frame over $TM \oplus T^*M$.
We will follow [3] to define a frame bundle in the ordinary case and then we will generalise it to the generalised geometry one.
Let us consider a manifold M endowed with a Riemannian metric g and a coordinate patch U_i of TM. It is possible to introduce a set of linearly independent vector fields $\{\hat{e}_a\}$, not related to any coordinate system.

Vielbeins

Firstly, we define non-coordinate basis by the concept of *vielbeins*. In a coordinate basis T_pM is spanned by $\{\partial/\partial x^\mu|_p\}$, where x^μ are the coordinates of a chart on a Riemannian manifold M containing the point p. As usual, the dual space T_p^*M is spanned by $\{dx^\mu\}$. Consider a "rotated" basis defined as follows

$$\hat{e}_a := {e_a}^\mu\ \partial_\mu \qquad \{{e_a}^\mu\} \in GL\,(d,\mathbb{R})^+\,. \tag{1.25}$$

Therefore we can state that the basis $\{\hat{e}_a\}$ is the *frame* of a basis obtained by rotating $\{\partial/\partial x^\mu\}$ by a $GL\,(d,\mathbb{R})$ transformation that preserves the orientation (*i.e.* up to a positive scaling factor, a $SL\,(d,\mathbb{R})$ transormation). This coefficients ${e_a}^\mu$ are called *vielbeins*, sometimes *vierbeins*, if we are in four dimensions. Furthermore, is required $\{\hat{e}_a\}$ to be an orthonormal frame with respect to the metric g,

$$g\,(\hat{e}_a,\hat{e}_b) = \delta_{ab}. \tag{1.26}$$

By linearity of the metric product (recall the metric is a $(0,2)$-tensor) we can write the previous expression as

$$g\,(\hat{e}_a,\hat{e}_b) = {e_a}^\mu {e_b}^\nu g\,(\partial_\mu,\partial_\nu)\,\delta_{ab} = {e_a}^\mu {e_b}^\nu g_{\mu\nu}\delta_{ab}.$$

If the metric is Lorentzian instead of Riemannian, δ_{ab} is replaced by η_{ab}. We can also express metric components in terms of (inverse) vielbeins,

$$g_{\mu\nu} = {e^a}_\mu {e^b}_\nu \delta_{ab}; \tag{1.27}$$

where ${e^a}_\mu$ is the inverse matrix of ${e_a}^\mu$, *i.e.* ${e^a}_\mu {e_a}^\nu = {\delta_\mu}^\nu$ and ${e^a}_\mu {e_b}^\mu = {\delta^a}_b$.
As for each basis any vector $v \in T_pM$ can be expressed in components with respect to the basis,

$$v = v^\mu \partial_\mu = v^a \hat{e}_a = v^a {e_a}^\mu \partial_\mu$$

and from these equalities we can write explicitly the relation between components

$$v^\mu = v^a {e_a}^\mu \qquad v^a = {e^a}_\mu v^\mu.$$

As expected, if the basis transforms with a matrix, components transform with the inverse matrix.
Consider now the dual basis $\{\xi^a\}$ defined by $(\xi^a,\hat{e}_b) = {\delta^a}_b$. We can see they are given

in terms of the dual coordinate basis as

$$\xi^a = e^a{}_\mu dx^\mu.$$

Thus, we can express the metric in terms of this basis as follows

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu = \delta_{ab} \xi^a \otimes \xi^b.$$

When we talk about *non-coordinate* basis we refer to the basis defined above, $\{\hat{e}_a\}$ and $\{\xi^a\}$. One of the most notable property of these basis is the fact that their Lie bracket is non-vanishing, in particular, if we take a non-coordinate basis defined as in (1.25) it satisfies the relation

$$[\hat{e}_a, \hat{e}_b]|_p = f_{ab}{}^c(p) \hat{e}_c|_p \tag{1.28}$$

where

$$f_{ab}{}^c(p) = e^c{}_\nu \left(e_a{}^\mu \partial_\mu e_b{}^\nu|_p - e_b{}^\mu \partial_\mu e_a{}^\nu|_p \right).$$

Note that the "structure constant" are **not** constant at all. They crucially depend on the point $p \in M$. The importance of this statement will become clear in what follows, in particular when we will analyse the generalised parallelisability in chapter 3.

The frame bundle

We have seen how a frame is defined on a point, or better, how is defined as a basis of the tangent space to the manifold in a point. Here, we want to extend our discussion to the whole bundle TM, defining the *frame bundle*.
As we have seen, the set $\{\hat{e}_a\}$ defines a local basis, said *local frame* on U_i and we can call $a = 1, \ldots, d$ the frame index.
In general, could be impossible to define these frames globally, as may not be possible to cover the whole manifold with a single chart.[4]
The *frame bundle* is the bundle associated with these basis vectors, defined as [27]

$$F := \bigcup_{p \in M} F_p, \tag{1.29}$$

where

$$F_p := \{(p, \{\hat{e}_a\}) \mid p \in M\}.$$

In other words, F_p is the set of all frames at $p \in M$. The frame bundle can be seen as a $GL(d, \mathbb{R})$-principal bundle, even if it seems to lack a proper group structure. This because

[4]In the particular case where a global frame is defined, the manifold is said *parallelizable*. This will be discussed later in chapter 3.

F_p is homeomorphic to $GL(d,\mathbb{R})$ as topological space, so we can associate naturally to the frame bundle a principal bundle with structure group $GL(d,\mathbb{R})$, we are not able to identify the identity element in a natural way. Now, the group $GL(d,\mathbb{R})$ acts freely and transitively on each fibre on the right - *i.e.* F_p is a principal homogeneous space for $GL(d,\mathbb{R})$ - to give another frame on the fibre. From this point of view we can see the action of the group $GL(d,\mathbb{R})$ as the way of changing frames keeping the point $p \in M$ fixed, so the change of frame is a tranformation on the fibre only. To be more explicit, two frames on F_p are related by

$$\hat{e}'_a = A_a{}^b \hat{e}_b, \qquad A_a{}^b \in GL\left(d,\mathbb{R}\right). \tag{1.30}$$

We can now proceed in two different directions, that turn out to be equivalent. The first choice we can operate is to restrict the transformations allowed to a subgroup $O(d) \subset GL(d,\mathbb{R})$, such that we take into account, in each point, just frames connected by an orthogonal transformation. We choose the group $O(d)$ since this group preserves the metric structure. We have a residual freedom: we can still choose which is the initial frame, the frame from which we start performing transformations to find the others. We can fix this choice by imposing a relation involving the metric ($O(d)$-invariant) between vectors of the frame, like

$$g\left(\hat{e}_a, \hat{e}_b\right) = \mathcal{A}_{ab}. \tag{1.31}$$

Imposing this relation is equivalent to fix the tensor $\mathcal{A}_{ab}$ as invariant tensor under $O(d)$ transformations.

An equivalent choice is to fix a relation the frame vectors must satisfy, one like the (1.31), or more conventionally

$$g\left(\hat{e}_a, \hat{e}_b\right) = \eta_{ab}, \tag{1.32}$$

where the matrix η is the usual Minkowski metric in a Lorentzian manifold (if g is Lorentzian), or the Kronecker delta in a Riemannian one. This choice and its invariant condition reduce the group of allowed transformation to the subgroup of $GL(d,\mathbb{R})$ that leaves the tensor η_{ab} invariant, thus the $O(d)$ group in the Riemannian case, which is the one we are taking into account. For a Lorentian metric the group would be $O(1,d-1)$, but nothing would be conceptually different, so we will analyse just the Riemannian situation.

The restriction to the $O(d)$ group implies $\{\hat{e}_a\}$ is now an orthonormal frame.

We have seen that imposing the invariance of the previous condition restricts the structure group $GL(d,\mathbb{R})$ to $O(d)$. This form an object known in mathematics as G-structure, where $G \subset GL(d,\mathbb{R})$ [28].

A G-structure is a principal subbundle of the tangent frame bundle $P \subset F$. For our concern the tangent frame bundle is the F in (1.29).

In our case, $G = O(d)$ and the correspondent subbundle is

$$P = \{(p, \{\hat{e}_a\}) \ , \forall p \in M \mid g(\hat{e}_a, \hat{e}_b) = \eta_{ab}\} \,. \tag{1.33}$$

In order to make this construction clearer take into account the following situation. Given an open cover $\{U_i\}$ of M and a point $p \in U_i \cap U_j$, any vector field $v_{(i)}$ on $TM\big|_{U_i}$ can be expressed in coordinates with respect to an (orthonormal) basis, thanks to the local frame on U_i, as $v_{(i)} = v_{(i)}^a \hat{e}_a$. Analogously for another vector field $v_{(j)}$ defined on $TM\big|_{U_j}$. Therefore, on each fibre of P, frames are connected by an $O(d)$ transformation,

$$\hat{e}'_a = \hat{e}_b \ A^b{}_a, \tag{1.34}$$

where we have denoted as $\{\hat{e}'_a\}$ the orthonormal frame on U_j, and $A \in O(d)$. The components in the two charts transform consequently,

$$v_{(j)}^a = \left(A^{-1}\right)^a{}_b \, v_{(i)}^b. \tag{1.35}$$

This fact hides something very deep, in fact, we have showed how imposing the invariance of the (1.33) reduces the structure group $GL(d)$ to its *maximally compact subgroup* $O(d)$. This is equivalent to the presence of an $O(d)$-structure on our manifold. This result descends from a fundamental theorem in the theory of Lie groups known as *Cartan-Iwasawa-Malcev theorem*, stating that every locally compact Lie group admits a maximally compact subgroup. This subgroup is unique up to conjugation, in the sense that given H and K, two maximally compact subgroups of G, there exists an element $g \in G$ such that $H \cong gKg^{-1}$. The uniqueness up to conjugation class corresponds to our freedom in choosing an invariant tensor in the (1.31), *i.e.* an inner product. Thanks to this property, we are allowed to talk about *the* maximally compact subgroup [29].
In light of this fact we can see the metric tensor as a representative of an equivalence class in $GL(d)/O(d)$, and for this reason we often refer to the metric structure as $O(d)$-structure. This new perspective provides a useful interpretation of the metric strucure that will be extremely useful in the construction of a generalised metric [9].

The generalised frame bundle

The definition of generalised frame bundle is straightforward from the generalisation of the frame bundle. Given a frame on E, that we name $\{\hat{E}_A\}$, satisfying the orthonormality condition with respect ot the natural inner product

$$\eta\left(\hat{E}_A, \hat{E}_B\right) \equiv \langle \hat{E}_A, \hat{E}_B \rangle = \eta_{AB} = \frac{1}{2}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}_{AB} \ , \tag{1.36}$$

we can define the *generalised frame bundle* as follows. The *frame bundle* is the bundle associated to these basis vectors. Points on the fibre (frames) are connected by $O(d,d)$ transformations. Conversely, all frames connected by $O(d,d)$ transformations to a given frame that satisfies (1.36) will satisfy it too. In other words, these frames form an $O(d,d)$-bundle, that we call *generalised frame bundle*,

$$F := \bigcup_{p\in M} \left\{ \left(p, \hat{E}_A\right) \mid p \in M,\ \eta\left(\hat{E}_A, \hat{E}_B\right) = \eta_{AB} \right\}. \tag{1.37}$$

Given the frame $\{\hat{e}_a\}$ for TM and $\{e^a\}$ for the cotangent bundle T^*M, we can made a particular choice of frame, that will be convenient in the further analysis. We introduce the *split frame* from frames on TM and T^*M such that we can keep track of the vector and form part of our generalised sections. Explicitly,

$$\hat{E}_A := \begin{cases} \begin{pmatrix} \hat{e}_a \\ -i_{\hat{e}_a} B \end{pmatrix} , & A = a\ ; \\ \begin{pmatrix} e^a \\ 0 \end{pmatrix} , & A = a + d\ . \end{cases} \tag{1.38}$$

Note that this is not simply the basis for $TM \oplus T^*M$ we defined in section 1.1.3, but the B-shift, the map that induces the splitting of E, is present in our definition of the split frame. This because we require the compatibility with patching rules defined above.

1.3.2 $O(d) \times O(d)$-structure

The fact that in the generalised case the structure group for the frame bundle is $O(d,d)$ and its maximally compact subgroup is $O(d) \times O(d)$ arises from various considerations we have done above. Now we wanto to investigate the meaning of introducing a metric and, as consequence, reducing $O(d,d)$ to $O(d) \times O(d)$, analogously as what has been seen before. This reduction, or splitting, has important consequences, and analysing it will let us recover an explicit representation for the generalised metric tensor G.

In order to motivate our way of introducing a generalised metric we can refer to figure 1.1. Consider a Riemannian manifold $(M,\ g)$ and observe that the metric g is completely described by its graph $C_+ = \{x + gx \mid x \in TM\} \subset TM \oplus T^*M$, where g is seen as a linear map from TM to T^*M. We are going to show that the restriction to G of the natural scalar product (1.1) is positive definite and allows us to define a generalised metric [16].

We can take as definition of generalised metric the definition **4.1.1** of [16].

Definition 1.6 (Generalised metric)**.** Let E be a generalised tangent bundle over M. A *generalised metric* C_+ is a positive definite sub-bundle of rank $d = \dim M$, that is, the restriction of the scalar product $\langle\cdot,\cdot\rangle$ to C_+ is positive definite.

The main goal of this section is to explore the meaning of the last definition.
We should now remark an important point: in mathematical literature like [4, 15, 16], the concept of generalised metric is referred to the sub-bundle on which the scalar product is restricted (as in definition 1.6), while in physics (as in [6, 9, 24]), and of course often also in this work, for generalised metric we mean the matrix G, describing the positive definite product.

Take into account the decomposition $O(d,d) \longrightarrow O(d)\times O(d)$, the generalised bundle E decomposes into two d-dimensional sub-bundles $C_+ \oplus C_-$[2, 24]. The sub-bundle C_+ is positive definite with respect to the inner product (1.1), while its orthogonal complement C_- is negative definited.
The presence of the natural metric η induces inner products on $C_\pm$ and this allows us to define a positive definite generalised metric G on E as

$$G(\cdot,\cdot) := \langle\cdot,\cdot\rangle\big|_{C_+} - \langle\cdot,\cdot\rangle\big|_{C_-}. \tag{1.39}$$

Since any generalised section which is made only by a pure vector field or a pure form has a zero norm with respect to the metric (1.2), we can state for instance $T^*M \cap C_\pm = \{0\}$. Analogously intersections between TM and $C_\pm$ are just made by the zero section, as represented in figure 1.1.
For these reasons we can define a map $h : TM \longrightarrow T^*M$ such that C_+ is the graphs of h, and C_- its orthogonal complement, and explicitly

$$C_+ = \{x + hx \mid x \in \Gamma(TM)\}. \tag{1.40}$$

The map h provides an isomorphism between TM and C_+.
It is clear that h is a section of $T^*M \otimes T^*M$ and hence can be written as $h = g + B$ exploiting the decomposition $T^*M\otimes T^*M \cong Sym^2T^*M \oplus \Lambda^2T^*M$, where $g \in Sym^2T^*M$ and $B \in \Lambda^2T^*M$. Thus we can write a general element $X_+ \in C_+$ as $X_+ = x + (B+g)\,x$, where as usual we denote with Bx the contraction i_xB. The orthogonality condition between C_+ and C_- force us to write $X_- \in C_-$ as $X_- = x + (B-g)\,x$ and so

$$C_- = \{x + (B-g)\,x \mid x \in \Gamma(TM)\}. \tag{1.41}$$

We can identify g with the familiar Riemannian metric thanks to the invariance of

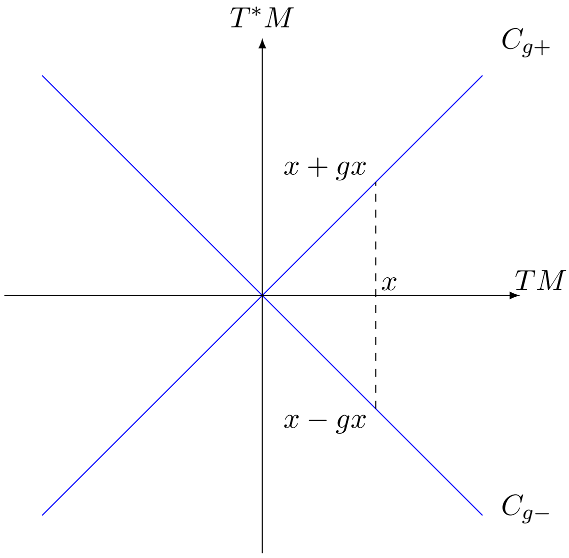

Figure 1.1: We can represent the splitting of E into the sub-bundles $C_+ \oplus C_-$ by the graph of a linear map $h : TM \longrightarrow T^*M$. Here is represented the particular case of a zero B field transformation.

the natural inner product under the shift by any 2-form B. In fact, take into account $X_+, Y_+ \in C_+$ and their inner product

$$\begin{aligned}\langle X_+, Y_+\rangle &= \langle x + i_x B + gx,\ y + i_y B + gy\rangle = \\ &= \langle x + gx,\ y + gy\rangle = \\ &= \frac{1}{2}\left(i_y gx + i_x gy\right) = g\left(x, y\right).\end{aligned}$$

As required $g(x, y)$ is the usual Riemannian (positive definite) inner product between two vector fields.

We have now all the tools to find the form of the generalised metric tensor G. Our strategy will be to use the definition of G in (1.39) in order to see how it acts on elements of C_+ and C_- and hence write an explicit matrix expression for G.

We can write a generalised vector $X \in E \cong C_+ \oplus C_-$ as $X = X_+ + X_-$, where $X_\pm \in C_\pm$. Thus we have the map G

$$\begin{aligned}G:\ E &\longrightarrow E^* \cong E \\ X &\longmapsto G(X) = G(X, \cdot)\end{aligned}$$

We denoted as $G(X)$ the generalised one-form $G(X, \cdot)$, but since the isomorphism $E^* \cong E$, as stated above in section 1.1.1, it can be thought as a vector. In fact, using the definition of G we can write $G(X)$ as

$$G(X) = G(X, \cdot) = \langle X_+ + X_-, \cdot\rangle\big|_{C_+} - \langle X_+ + X_-, \cdot\rangle\big|_{C_-} = X_+ - X_-\ . \tag{1.42}$$

From the last expression it is clear that $G^2 = \mathbb{1}$, and that $C_\pm$ are the eigenspaces relative to eigenvalues ± 1 of G. Consider firstly the simpler case represented in figure 1.1, with $B = 0$. The map h now is just the Riemannian metric g on the manifold M, and the two sub-bundles are

$$C_{g\pm} := \{x \pm gx \mid x \in TM\} \ .$$

A generic vector in $C_{g\pm}$ can be written as $X_{g\pm} = x \pm gx$. In this particular case $2x = X_{g+} + X_{g-}$, thus, we are now allowed to write

$$2G(x) = X_{g+} - X_{g-} = 2g(x),$$

and since $G^2 = \mathbb{1}$, it holds

$$2G^2(x) = 2G(g(x)) = X_{g+} + X_{g-} = 2x.$$

In this basis, the simplest form the matrix G can take is

$$G_g = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \ .$$

Now we want to reintroduce the B field we have ignored so far. Recall that

$$e^B X_{g\pm} = (x \pm gx + Bx) = X_\pm$$

and also that

$$G\left(X_\pm\right) = \pm X_\pm. \tag{1.43}$$

Using the previous relations and applying the B transformation to $X_{g\pm}$, we can obtain a matrix representation for G as follows,

$$\begin{aligned} X_\pm = e^B X_{g+} &= \pm e^B G_g X_{g+} = \\ &= \pm e^B G_g e^{-B} e^B X_{g+} = \\ &= \pm e^B G_g e^{-B} X_+ \ . \end{aligned}$$

This is true if and only if

$$G = e^{-B} G_g e^B = \begin{pmatrix} g^{-1}B & g^{-1} \\ g - Bg^{-1}B & -Bg^{-1} \end{pmatrix}. \tag{1.44}$$

What we would like to emphasise here is that we have written the generalised metric G as an object corresponding to the group structure reduction $O(d,d) \longrightarrow O(d) \times O(d)$, only in terms of g and B. This is a remarkable result of this work, since we obtained a single geometrical object encoding both the fields appearing in the type IIA and type IIB supergravity NS-NS sector, as already shown in [6, 24, 30].

We have seen in the previous section how the metric g in ordinary differential geometry can be thought as a class representative of the coset space $GL(d)/O(d)$, whereas in generalised geometry we can state that G parametrises the homogeneous space

$$O(d,d)/\left(O(d)\times O(d)\right). \tag{1.45}$$

Actually we have not proved that the 2-form B that appears in the generalised metric's construction is the B-field defined as the splitting map in the section 1.2.2. We can prove this considering the local expression of C_+ as a graph of the map $x+g_{(i)}x+B_{(i)}x$, defined on the chart U_i, and look at its behaviour under the transition to a different coordinate chart U_j. The expression in the chart U_j can hence be written as

$$x+g_{(j)}x+B_{(j)}x = x+g_{(i)}x+B_{(i)}x+i_x\mathrm{d}\Lambda_{(ij)}.$$

Equating the symmetric and antisymmetric parts separately we find that $g_{(i)}=g_{(j)}$, defining a proper Riemannian metric,[5] while $B_{(j)}=B_{(i)}+\mathrm{d}\Lambda_{(ij)}$, that is precisely the request for a connection structure on a gerbe [20], or, as we have seen before, it is the patching prescription for the B-field in (1.20) [16].
There is an alternative way to construct the matrix representation (1.44) for the generalised metric, following [9] and using generalised frames encoding both the geometrical objects g and B and from these reconstruct G.
We are interested in finding, into the generalised frame bundle F, two set of frames $\{\hat{E}^+_a\}$ and $\{\hat{E}^-_{\bar{a}}\}$ $(a,\bar{a}=1,\dots,d)$ satisfying

$$\langle \hat{E}^+_a,\ \hat{E}^+_b\rangle=\delta_{ab}\ , \qquad \langle \hat{E}^-_{\bar{a}},\ \hat{E}^-_{\bar{b}}\rangle=-\delta_{\bar{a}\bar{b}}\ , \qquad \langle \hat{E}^+_a,\ \hat{E}^-_{\bar{b}}\rangle=0\ . \tag{1.46}$$

Here the $O(d)\times O(d)$ symmetry is explicit and each $O(d)$ factor acts on a different set of frames, for this reason sometimes the group symmetry is denoted as $O(d)^+\times O(d)^-$. Explicitly, a generic element in $O(d)^+\times O(d)^-$ can be written as

$$\Theta=\begin{pmatrix}\vartheta^+ & 0\\ 0 & \vartheta^-\end{pmatrix}\ ,$$

and acts on $\hat{E}^\pm$ such that

$$\begin{cases}\hat{E}^+_a\to\vartheta^+_a{}^b\hat{E}^+_b\\ \hat{E}^-_{\bar{a}}\to\vartheta^-_{\bar{a}}{}^{\bar{b}}\hat{E}^-_{\bar{b}}\end{cases}$$

An equivalent definition is via the product structure G in the following projectors on $C_\pm$,

$$\Pi_\pm:=\frac{1}{2}\left(\mathbb{1}\pm\mathcal{P}\right), \tag{1.47}$$

[5] Again, we are not considering the $GL(d,\mathbb{R})$ transformations.

with $\mathcal{P}^2 = \mathbb{1}$ and $\mathcal{P}\hat{E}^{\pm} = \pm\hat{E}^{\pm}$, we can compare with (1.43), since the resemblance is evident at this stage.
This point of view hides something very deep, since here one can note a similarity with the so called almost complex structures [31] in ordinary differential geometry, or with pre-symplectic structures [32], this was in infact the starting point for Hitchins in order to generalise to generalised complex structures. The deepness of the generalised geometry construction is that from this point of view all these kinds of structures seems to have a similar description.

In order to induce the inner product on $C_\pm$ we require a further condition on the operator $\mathcal{P}$,

$$\langle \mathcal{P}V,\ \mathcal{P}W\rangle = \langle V,\ W\rangle \quad \forall V, W \in \Gamma(E)\ ,$$

or equivalently $\mathcal{P}^T\eta\mathcal{P} = \eta$.
Now we can construct an analogous of the split frame defined in (1.38) from frames on $C_\pm$ rather than from ones on TM and T^*M.
Take on $C_\pm$ the frames $\{\hat{e}^+_a\}$ and $\{\hat{e}^-_{\bar{a}}\}$ and their duals $\{e^+_a\}$, $\{e^-_{\bar{a}}\}$ respectively. Then a solution to the system of costrains (1.46) can be written as

$$E^+_a = \left(\hat{e}^+_a - i_{\hat{e}^+_a}B\right) + e^+_a\ , \tag{1.48}$$

$$E^-_{\bar{a}} = \left(\hat{e}^-_{\bar{a}} - i_{\hat{e}^-_{\bar{a}}}B\right) + e^-_{\bar{a}}\ . \tag{1.49}$$

Thus, the generalised metric takes the form

$$G = \delta^{ab}E^+_a \otimes E^+_b + \delta^{\bar{a}\bar{b}}E^-_{\bar{a}} \otimes E^-_{\bar{b}}\ , \tag{1.50}$$

and expressing the previous quantity in the coordinate basis of $TM \oplus T^*M$ and imposing $\mathcal{P}^T\eta\mathcal{P} = \eta$, we find

$$G_{AB} = \eta_{AC}\mathcal{P}^C{}_B\ ; \qquad \text{where} \quad P^A{}_B = \begin{pmatrix} g^{-1}B & g^{-1} \\ g - Bg^{-1}B & -Bg^{-1} \end{pmatrix}^A{}_B\ ,$$

that is precisely what we found above in (1.44), providing the equivalence of the two approaches described.

1.4 An aside: why $TM \oplus T^*M$?

As stated previously, we have constructed an object that encodes both the field g and B, appearing in the spectrum of the NS-NS sector of type II supergravity. Actually, we miss the dilaton ϕ, but its inclusion is straightforward as showed by Waldram et al. in [6, 9]. This can be considered a partial success of our approach, in fact, one of the main aims of our work was to understand the geometrical meaning of the fields appearing in the supergravity spectrum, so the generalised metric is the kind of object we were looking for.

We should remark that G does not give us any information about the dynamics of the fields, we need the supergravity action for this, but now we may be able (with some further effort that we are going to face in the next section) to write the action of this sector in a very elegant way, analogous to the Einstein's general relativity, *i.e.* as a pure metric theory.

Here, as conclusion of this part, we want to give an *a priori* motivation that could bring a physicist to study generalised geometry. Consider the supergravity action for the NS-NS sector [9, 33],

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^{10}x \sqrt{-g} e^{-2\phi} \left(\mathcal{R} + 4\partial_\mu \phi \partial^\mu \phi - \frac{1}{12} H^2 \right) \ , \tag{1.51}$$

this exhibits two important symmetries, the *diffeomorphism invariance* and the *gauge invariance*. The first one comes from the fact that the supergravity is an extension of the Einstein's general relativity, the second from the promotion of supersymmetry to a gauge symmetry.

Diffeomorphisms are parametrised by vector fields $v \in \Gamma(TM)$, while gauge transformations by one forms $\lambda \in \Gamma(T^*M)$. To make this evident let us express the infinitesimal variations of the fields by,

$$\delta_v g = \mathcal{L}_v g \ , \qquad \delta_v \phi = \mathcal{L}_v \phi \ , \qquad \delta_v B = \mathcal{L}_v B + \mathrm{d}\lambda \ .$$

From these it is clear that under a gauge transformation we have $B \to B + \mathrm{d}\lambda$, and so that gauge transformations are parametrised by λ. It is also evident that the vector v parametrises the diffeomorphisms, written by taking the Lie derivative along v.

Thus, this suggests to consider a generalised tangent bundle that could accomodate in one single object the vector field and the one-form. This can be seen as an euristic motivation to introduce the generalised tangent bundle $TM \oplus T^*M$.

Actually, other extensions can be considered, to encode also other symmetries of more general actions. For example in [34] Hull introduced an exceptional generalised geometry, then further developed by Pacheco, Waldram et al. in [26, 35]. This approach considers a generalised tangent bundle with an $E_{d(d)} \times \mathbb{R}^+$ structure to incorporate symmetries of the M-theory. We are not treating these arguments in this thesis, but their existence could be a motivation to further investigate the relations between generalised geometry and string theory.

Chapter 2

Generalised Connection, Torsion, Curvature

Once the concept of metric has been inctroduced we can build the generalised analogous of all the related objects that we know in ordinary differential geometry. In this chapter we introduce a generalised notion of connection and a generalised torsion tensor. Equipped with this tools, we are going to define a generalisation of the Riemannian curvature tensor and, by its contraction, the generalised parallel of Ricci tensor and scalar. We are going to take follow [6] to show the key results of this construction.

The notion of connection on Courant algebroids was first introduced by Xu, Stienon et al. in [36] and Gualtieri and Cavalcanti in [17] and plays a very important role in this geometric formulation of supergravity.
We are going to define the main objects by following [9] principally. Then, found the connection components in coordinates related to the frames introduced in section 1.3.1, we will define the torsion via the torsion map as in ordinary differential geometry. In the end, after discussing the generalised analogous of a Levi-Civita connection and its non-unicity we will construct an expression for the curvature map and write a "*generalised Einstein equation*", that, we will show, encodes the field equations for the type II NS-NS sector of supergravity.

2.1 Generalised connections

2.1.1 Definition and properties

The *connection* in differential geometry is an object that specifies how tensors are transported along a given curve on a manifold, and how this transport modifies them. In the generalised enviroment we want to define an analogous notion in this sense. As usual we begin considering vector fields, since the generalisation is straightforward.

Definition 2.1 (Generalised connection)**.** A generalised (affine) connection D is a map

$$\begin{aligned} D\colon\ \Gamma(E)\times\Gamma(E) &\longrightarrow \Gamma(E)\\ (V,W) &\longmapsto D_V(W) \end{aligned}$$

such that satisfies the following properties for all $U,\ V,\ W\in\Gamma(E)$ and $f\in\mathcal{F}(M)$,

i. $D_V(W+U)=D_V(W)+D_V(U)$;

ii. $D_{V+U}(W)=D_V(W)+D_U(W)$;

iii. $D_{fV}(W)=fD_V(W)$;

iv. $D_V(fW)=fD_V(W)+V[f]\,W$.

As usual we are also interested in a coordinate expression, therefore consider a coordinate basis $\{\hat{E}_M\}$. We can write the components of the connection as follows

$$D_{\hat{E}_M}\hat{E}_N := \Gamma_M{}^K{}_N\hat{E}_K\ . \tag{2.1}$$

We will refer to $\Gamma_M{}^K{}_N$ as *generalised connection components.*

2.1.2 Spin connection

We could also define a generalisation of the spin connection, defined using vielbeins (see [37]) as follows. First of all, recall how the *spin connection* is defined in conventional differential geometry, consider a non coordinate basis $\{\hat{e}_a\}$ defined via vielbeins as in 1.3.1 from a coordinate basis $\{\partial_\mu\}$, we can define a connection $\nabla^{(s)}_\mu$ acting on a vector v^a as in the following expression[1] [38]

$$\nabla^{(s)}_\mu v^a=\partial_\mu v^a+\omega_\mu{}^a{}_b v^b\ . \tag{2.2}$$

[1]Note the different nature of the indeces on ∇_μ and v^a.

Note that the presence of the metric is crucial, since it induces an $O(d)$-subbundle and hence a *spinor bundle*, where defining the covariant derivative of spinors is allowed. Inserting the expression of vector $v^a = e^a{}_\mu v^\mu$ in the previous formula, we find a precise form for the spin connection components $\omega_\mu{}^a{}_b$,

$$\omega_\mu{}^a{}_b = e^a{}_\nu\, \omega^\nu{}_{b\mu}$$

where,

$$\omega^\nu{}_{a\mu} := \nabla_\mu e^\nu{}_b = \partial_\mu e^\nu{}_b + \Gamma^\nu_{\mu\rho} e^\rho{}_b \ ,$$

and ∇_μ is the familiar affine connection with components $\Gamma^\nu_{\mu\rho}$. From this in supergravity is often defined a *total covariant derivative* $\mathcal{D}_\mu$ acting on objects with mixed frame indeces as

$$\mathcal{D}_\mu A^\nu{}_a = \partial_\mu A^\nu{}_a + \Gamma^\nu_{\mu\rho} A^\rho{}_a - \omega_\mu{}^b{}_a A^\nu{}_b \ .$$

Moving into a non-coordinate frame $\{\hat{E}_A\}$, in the generalised case we define a generalised connection as in [9] by the expression

$$D_M V^A = \partial_M V^A + \Omega_M{}^A{}_B V^B \ , \tag{2.3}$$

where we indicated with M the index for the coordinate generalised basis $\{\partial_M\}$ as defined in 1.1.3.

The operator D defined in (2.3) is a differential operator (it satisfies the Leibnitz rule for the product of two tensor fields) and its action can be extended, as usual, to a generic tensor field like $T \in \Gamma\left(E^{\otimes r}\right)$ as

$$D_M T^{A_1 \ldots A_r} = \partial_M T^{A_1 \ldots A_r} + \Omega_M{}^{A_1}{}_B T^{B \ldots A_r} + \ldots + \Omega_M{}^{A_r}{}_B T^{A_1 \ldots B} \ .$$

In order to find some constraints on the connection components, consider the equation (1.46) that gives us the expression of the $O(d,d)$ metric in terms of the frame basis,

$$\eta_{AB} = \eta_{MN} \hat{E}_A{}^M \hat{E}_B{}^N \ . \tag{2.4}$$

We now impose metric compatibility condition of the connection. This implies the antisymmetry of connection components as one explicitly see,

$$D_M \eta_{AB} = \partial_M \eta_{AB} - \Omega_M{}^C{}_A \eta_{CB} - \Omega_M{}^C{}_B \eta_{AC} \ .$$

Equating the last expression to zero, finally we get the antisymmetry condition

$$\Omega_{MAB} = -\Omega_{MBA} \ . \tag{2.5}$$

2.1.3 Lifting of an ordinary connection ∇

There is also the possibility of defining a generalised connection given an usual connection ∇ and a split frame like (1.38), by lifting the action of ∇ on the components (with respect to the split frame) to an action of a new object D^{∇}_{M} on E. The object such defined can be proved by direct check to be a generalised connection.
Explicitly, take a generalised vector field $V \in \Gamma(E)$, write it in components as

$$V = V^A \hat{E}_A = v^a \hat{E}_a + \lambda_a E^a \ ,$$

then the generalised connection defined (or, from a certain point of view, "induced") by ∇ takes the form

$$\left(D^{\nabla}_{M} V\right) = \left(D^{\nabla}_{M} V^A\right) \hat{E}_A = \begin{cases} \left(\nabla_\mu v^a\right) \hat{E}_a + \left(\nabla_\mu \lambda_a\right) E^a & M = \mu \\ 0 & M = \mu + d \end{cases} . \tag{2.6}$$

If we want we can express this in a more schematic way, by a commutative diagram,

$$\begin{tikzcd} \Gamma(E) \times \Gamma(E) \arrow[r, "D^{\nabla}"] \arrow[d, "B"] & \Gamma(E) \arrow[d, "B"] \\ \Gamma(TM \oplus T^*M) \times \Gamma(TM \oplus T^*M) \arrow[r, "\nabla"] & \Gamma(TM \oplus T^*M) \end{tikzcd} \tag{2.7}$$

where has been made evident that the B-field, once again, plays a crucial role in definitions of generalised objects. In this case we can equivalently define a generalised connection D^{∇} by the expression

$$D^{\nabla}_{V} W = e^B \left(\nabla_{\pi(V)} e^{-B} W\right) = \nabla_v w + \nabla_v \tilde{\zeta} - i_{\nabla_v w} B \ . \tag{2.8}$$

This formula represents nothing else than the commutative diagram (2.7) with a choice of B-splitting, and with $\tilde{\zeta}$ we denoted the form component on $TM \oplus T^*M$ after the action of the e^{-B} map.

2.2 Generalised torsion

Going on with our construction of generelised objects, we want to define the concept of *generalised torsion*. One way to do it is following a generalisation of the procedure to that showed in [3] for the ordinary differential geometry. Firstly, defining a torsion map of a general connection ∇, $T(x, y) := \nabla_x y - \nabla_y x - [x, y]$ and then, acting on a basis, obtaining a torsion tensor. However, though this approach is very elegant, it clashes with the fact that in generalised geometry we have two notions of Lie brackets. To solve this we note that the torsion map has a nice expression in terms of the Lie derivative given by

$$T(x,y) = x^\mu \nabla_\mu y - y^\mu \nabla_\mu x - x^\mu \partial_\mu y + y^\mu \partial_\mu x := \mathcal{L}^\nabla_x y - \mathcal{L}_x y \ , \tag{2.9}$$

where we denoted as $\mathcal{L}^\nabla$ the Lie derivative expression with the substitution of the ordinary derivative with the covariant one, $\partial \to \nabla$ [9].
Going in components, that is analysing the torsion tensor, we can focus our attention on the adjoint action of the group $GL(d)$,

$$x^\mu T^\alpha{}_{\mu\nu} y^\nu = \left(T_x\right)^\alpha{}_\nu y^\nu \ .$$

In the last expression the matrix $\left(T_x\right)^\alpha{}_\nu$ lives in the adjoint rep of $\mathfrak{gl}\left(d, \mathbb{R}\right)$ and, by dint of these observations, we can generalise the torsion map defining its action on any tensor, for instance

$$\begin{aligned} T_x A^{\alpha_1\ldots\alpha_r}{}_{\beta_1\ldots\beta_s} &= \mathcal{L}^\nabla_x A^{\alpha_1\ldots\alpha_r}{}_{\beta_1\ldots\beta_s} - \mathcal{L}_x A^{\alpha_1\ldots\alpha_r}{}_{\beta_1\ldots\beta_s} = \\ &= A^{\mu\alpha_2\ldots\alpha_r}{}_{\beta_1\ldots\beta_s} x^\nu \left(\Gamma^{\alpha_1}_{\nu\mu} - \Gamma^{\alpha_1}_{\mu\nu}\right) + \ldots + \\ &- A^{\alpha_1\ldots\alpha_r}{}_{\nu\beta_2\ldots\beta_s} x^\nu \left(\Gamma^\mu_{\nu\beta_1} - \Gamma^\mu_{\beta_1\nu}\right) - \ldots \end{aligned}$$

where $\Gamma^\alpha_{\mu\nu}$ are the connection components.
Here we can recognise the adjoint $\mathfrak{gl}(d)$ matrices $\left(T_x\right)^\alpha{}_\mu = x^\nu \left(\Gamma^\alpha_{\nu\mu} - \Gamma^\alpha_{\mu\nu}\right)$ acting on each tensor component.

In the same way, we generalise the torsion tensor by the Dorfman derivative, rather than by a generalised torsion map.

Definition 2.2 (Generalised Torsion)**.** The generalised torsion of a generalised connection D is defined as the following map,

$$\begin{aligned} T \colon \Gamma\left(E\right) \times \Gamma\left(E\right) &\longrightarrow \Gamma\left(E\right) \\ (V,W) &\longmapsto T(V)W := L^D_V W - L_V W \end{aligned}$$

where $L^D_V W$ is the Dorfman derivative in (1.10) with the prescription $\partial \to D$.

Given a basis we can write down an expression in components of the quantity $T(V)W$ as follows,

$$\begin{aligned} T(V)W &= V^R W_N \left(\eta^{NK} \Gamma_R{}^M{}_K + \eta^{MK} \Gamma_K{}^N{}_R - \eta^{NK} \Gamma_K{}^M{}_R\right) \hat{E}_M = \\ &= T_R{}^M{}_K V^R W^K \hat{E}_M \ . \end{aligned}$$

Analogously to what we have seen above, we can apply the last definition to a generalised tensor field $A \in \Gamma(E^{\otimes s})$

$$T(V)A := L^D_V A - L_V A \; .$$

Therefore, from this point of view, we can think T as a torsion map from the sections of E to $adj(F)$, where $adj(F)$ is the *adjoint bundle* related to the principal bundle F in (1.37).

2.3 Levi-Civita connections

Once a notion of torsion has been introduced, one could wonder if exists a (unique) generalised parallel of the *Levi-Civita connection*, that is, a connection metric compatible and torsion-free. The answer of this question, as we are going to show, is positive. However, in the generalised case we are not able to find a unique connection both metric compatible and torsion-free.
Moreover, we should clarify what "*metric compatible*" and "*torsion-free*" mean in the generalised environment. The following section is aimed to explore these concepts and their implications.

Definition 2.3 (Generalised Levi-Civita connection)**.** Given a differentiable manifold M endowed with a generalised metric structure $G = \eta \cdot \mathcal{P}$, we define a *generalised Levi-Civita connection* as a generalised connection that in addition satisfies the following properties

i) $O(d,d)$-structure compatibility $D\eta = 0$;

ii) $O(d) \times O(d)$-structure compatibility $D\mathcal{P} = 0$;

iii) Torsion-free $T^D = 0$.

It is immediate to show that conditions i and ii imply $D(G) = 0$. Furthermore, the first condition also restricts the connection components to be antisymmetric in the last two indeces, *i.e.*

$$\Gamma_{MNK} = \eta_{NP}\Gamma_M{}^N{}_K = \Gamma_{M[NK]} \; .$$

The third condition, that express the torsionless of the connection D constrains further the connection components

$$T_{MNK} = \eta_{NP}T_M{}^N{}_K = 3\Gamma_{[MNK]} = 0 \; ;$$

from which is clear that $T \in \Lambda^3 E$.
In order to understand better the meaning of the definition 2.3, we can investigate the conditions i)-iii) in a different coordinate system, like an $O(d) \times O(d)$ frame like that defined in (1.48),

$$\begin{aligned} D_M \hat{E}^+_a &= \Omega_M{}^b{}_a \hat{E}^+_b + \Omega_M{}^{\bar{b}}{}_a \hat{E}^-_{\bar{b}} \\ D_M \hat{E}^-_{\bar{a}} &= \Omega_M{}^b{}_{\bar{a}} \hat{E}^+_b + \Omega_M{}^{\bar{b}}{}_{\bar{a}} \hat{E}^-_{\bar{b}} \ . \end{aligned}$$

We impose the $O(d) \times O(d)$ compatibility and we find a simple condition on the coefficients – dropping the M index – we can write $\Omega^{\bar{b}}{}_a = \Omega^b{}_{\bar{a}} = 0$, that means that the action of the connection does not mix C_+ and C_- bundles components, that is, it does not map C_+ into C_- or viceversa. Finally, imposing condition i) we obtain an antisymmetric constraint on the components, $\Omega_{ab} = \Omega_{[ab]}$, where we raise/lower indices by the $O(d)$ metric. The same holds also for the C_- components.

Moreover, we can be a bit more precise about the meaning of compatibility of a connection with a G-structure in general. We are going to point out some notions widely studied in mathematics, for instance in [18, 27], so we will be very brief and synthetic.
We have seen above in section 1.3.1 that a G-structure is a principal sub-bundle $P \subset F$ with fibre $G \subset \mathcal{G}$, where $\mathcal{G}$ is the group structure of the principal bundle F (in all this work $\mathcal{G} = GL(d)$). Defining a metric, we take $G = O(d)$ and P as the bundle of orthonormal frames in (1.33). Points in P, we have seen are connected by $O(d)$ transormations and g, at each point, defines a class in $GL(d)/O(d)$.
Now, focus our attention on the fact that a general connection ∇ is compatible with a G-structure if and only if the corresponding connection on the principal bundle F, restricted to the sub-bundle P defining the G-structure is still a connection on it [27]. This means that once a frame $\{\hat{e}_a\}$ is given is possible to find a set of connection one-forms $\omega^a{}_b$ defined implicitly by

$$\nabla_\mu \hat{e}_a = \omega_\mu{}^b{}_a \ \hat{e}_b \ .$$

Note that these forms take values in the adjoint representation of the group G.
For an $O(d)$-structure defining a metric g, the former condition is equivalent to $\nabla g = 0$, that is indeed always named *metric compatibility* of the connection.
The existence of a connection G-structure compatible and torsion-free is not obvious and in general can further restrict the structure. However, if $G = O(d)$ and therefore we have a metric, this does not put any restriction on g. In this case the torsion free, compatible connection exists unique and is called *Levi-Civita*.

Now we are going to prove the existence of a Levi-Civita connection in the generalised case (in the sense of definition 2.3). The price paid for this generalisation is the loss of uniqueness of this kind of object in this context. the prove of this important result was found firstly by Waldram et al. in [6, 9] and our proof follows that one quite closely.

Theorem 2.4 (Existence of a Levi-Civita connection)**.** *Given an $O(d)\times O(d)$-structure, there always exists a torsion-free, structure compatible generalised connection D.*

Proof. We are going to prove the theorem through a constructive procedure. We are going to impose the constraints given by definition 2.3 on the most general connection possible. We are also going to use what we have seen about the lifting of an ordinary Levi-Civita connection ∇.

Given ∇, ordinary Levi-Civita connection, acting on a split frame, like (1.48), as

$$\begin{aligned}\nabla_\mu \hat{e}^+_a &= \omega_{\mu}^{+b}{}_a \hat{e}_b\\ \nabla_\mu \hat{e}^-_{\bar{a}} &= \omega_{\mu}^{-\bar{b}}{}_{\bar{a}} \hat{e}^-_{\bar{b}}\ .\end{aligned}$$

In the previous section 2.1.3 we have shown how this defines a generalised connection D^∇. Furthermore, we require the $O(d)\times O(d)$ compatibility, hence its action on the frame $\{\hat{E}^+_a\}$ and $\{\hat{E}^-_{\bar{a}}\}$ is constrained to be

$$\begin{aligned} D^\nabla_M \hat{E}^+_a &= \begin{cases} \omega_{\mu}^{+b}{}_a \hat{E}^+_b & \text{for } M=\mu \\ 0 & \text{for } M=\mu+d \end{cases}\\ D^\nabla_M \hat{E}^-_{\bar{a}} &= \begin{cases} \omega_{\mu}^{-\bar{b}}{}_{\bar{a}} \hat{E}^-_{\bar{b}} & \text{for } M=\mu \\ 0 & \text{for } M=\mu+d \end{cases}\ . \end{aligned} \tag{2.10}$$

where, again dropping the index μ

$$\Omega^b{}_a = \omega^{+b}{}_a\ , \qquad \Omega^{\bar{b}}{}_{\bar{a}} = \omega^{-\bar{b}}{}_{\bar{a}}\ , \qquad \Omega^{\bar{b}}{}_a = \Omega^b{}_{\bar{a}} = 0\ .$$

Such that we find D^∇ compatible with the generalised metric projector $\mathcal{P}$. Moreover, ∇ is a Levi-Civita connection, therefore its components $\omega^\pm_{ab}$ are antisymmetric and this makes D^∇ also compatible with η.

We are left with the condition *iii*) of definition 2.3, the torsionless condition. We can calculate the torsion of the connection explicitly, as done in [9], to find $T=-4H$. Hence, the connection D^∇ is not torsion-free. Nevertheless, not everything is lost. A generic connection can be written as

$$D_M = D^\nabla_M + \Sigma_M\ .$$

We can easily impose the $O(d) \times O(d)$ compatibility by choosing a Σ such that

$$\Sigma_{Mab} = -\Sigma_{Mba} \ , \qquad \Sigma_{M\bar{a}\bar{b}} = -\Sigma_{M\bar{b}\bar{a}} \ , \qquad \Sigma_{M\ a}^{\ \ \bar{b}} = \Sigma_{M\ \bar{a}}^{\ \ b} = 0 \ .$$

Now, in order to make D torsionless, we impose $T\left(D^{\nabla}\right) = -T\left(\Sigma\right)$ wich implies

$$3\Sigma_{[ABC]} = -4H_{ABC} \ . \tag{2.11}$$

To sum up, finally, we have explicitly constructed a generalised connection D that is torsion-free and structure compatible, as required by definition 2.3. This proves the theorem. □

The quantities H_{ABC} we indicated above are the components of the 3-form $H = \mathrm{d}B$ in the frame $\{\hat{E}^{\pm}_{A}\}$, under the embedding $\Lambda^3 T^*M \hookrightarrow \Lambda^3 E$, that decomposes as

$$\Lambda^3 T^*M \hookrightarrow \Lambda^3 E \cong \Lambda^3 C_+ \oplus 3\left(\Lambda^2 C_+ \oplus \Lambda C_-\right) \oplus 3\left(\Lambda C_+ \oplus \Lambda^2 C_-\right) \oplus \Lambda^3 C_- \ .$$

It is a very interesting point to notice that the coefficients emerging in this decomposition are the same that arise in supergravity, making us more confident about the usefulness of this approach.

Another apparently surprising result is the fact that the generalised torsion tensor T is an element in $\Lambda^3 E$ and not, as one might expect, in $\Lambda^2 E \otimes E$. This comes from the isomorphism $E \cong E^*$ we pointed out in the section 1.1.1 and from the embeddings $TM \hookrightarrow E$ and $T^*M \hookrightarrow E$.

However, the generalised connection D constructed above is not unique. In fact, let us analyse more closely the equation (2.11). In the split frame basis $\{\hat{e}^{\pm}\}$, it appears as the set of conditions

$$\begin{aligned}
\Sigma_{[abc]} &= -\frac{1}{6}H_{abc} \ , \qquad & \Sigma_{\bar{a}bc} &= -\frac{1}{2}H_{\bar{a}bc} \ , \qquad & \Sigma_{a\ b}^{\ a} &= 0 \ , \\
\Sigma_{[\bar{a}\bar{b}\bar{c}]} &= +\frac{1}{6}H_{\bar{a}\bar{b}\bar{c}} \ , \qquad & \Sigma_{a\bar{b}\bar{c}} &= +\frac{1}{2}H_{a\bar{b}\bar{c}} \ , \qquad & \Sigma_{\bar{a}\ \bar{b}}^{\ \bar{a}} &= 0 \ .
\end{aligned}$$

These do not determine Σ completely, in fact some components may be written as

$$\Sigma_{a\ c}^{\ b} = \tilde{\Sigma}_{a\ c}^{\ b} + A_{a\ c}^{\ b} \qquad \Sigma_{\bar{a}\ \bar{c}}^{\ \bar{b}} = \tilde{\Sigma}_{\bar{a}\ \bar{c}}^{\ \bar{b}} + A_{\bar{a}\ \bar{c}}^{\ \bar{b}} \ ,$$

where $\Sigma_{[abc]} = \tilde{\Sigma}_{[abc]}$ and $\Sigma_{[\bar{a}\bar{b}\bar{c}]} = \tilde{\Sigma}_{[\bar{a}\bar{b}\bar{c}]}$, and so $A_{[abc]} = A_{[\bar{a}\bar{b}\bar{c}]} = 0$ and $A_{a\ b}^{\ a} = A_{\bar{a}\ \bar{b}}^{\ \bar{a}} = 0$, such that A components do not contribute to the torsion. We can state, equivalently, that A lies in a representation of $O(d) \times O(d)$ such that its contribution to the torsion is zero. The antisymmetry on the last two indeces is inherited from Σ, this imposes some constraints on A, however, some of its components are uncostrained quantities

and so arbitrary. This freedom in choosing some components of A encodes the lack of uniqueness for a Levi-Civita connection we stated above.
Finally, given a generalised section $V = v_+^a \hat{E}_a^+ + v_-^{\bar{a}} \hat{E}_{\bar{a}}^-$, in the frame coordinates, one can write explicitly how a Levi-Civita generalised covariant derivative is expressed when it acts on V,

$$\begin{aligned} D_a v_+^b &= \nabla_a v_+^b - \frac{1}{6} H_a{}^b{}_c v_+^c + A_a{}^b{}_c v_+^c \ , \\ D_{\bar{a}} v_+^b &= \nabla_{\bar{a}} v_+^b - \frac{1}{2} H_{\bar{a}}{}^b{}_c v_+^c \ , \\ D_a v_-^{\bar{b}} &= \nabla_a v_-^{\bar{b}} + \frac{1}{2} H_a{}^{\bar{b}}{}_{\bar{c}} v_-^{\bar{c}} \ , \\ D_{\bar{a}} v_-^{\bar{b}} &= \nabla_{\bar{a}} v_-^{\bar{b}} + \frac{1}{6} H_{\bar{a}}{}^{\bar{b}}{}_{\bar{c}} v_-^{\bar{c}} + A_{\bar{a}}{}^{\bar{b}}{}_{\bar{c}} v_-^{\bar{c}} \ , \end{aligned} \tag{2.12}$$

and the non-uniqueness is explicitly represented by the undetermined coefficients A. Although this, it is still possible define some covariant operators, unambiguously by contractions that make the A part null. We are going to discuss this briefly in the next section.

2.3.1 Unique supergravity spinor equations

The fact that the Levi-Civita connection in the generalised case is not unique might raise doubts about the validity of our approach, since ambiguities could come and applications to supergravity – in which expressions must be unambiguos – would be prevented. However, we can show how is possible to find equations independent on the particular D chosen (*i.e.* non depending on A), and so constructing equations for a pair of chiral spinor fields $\varsigma^\pm$, like in [6].
This is also the way followed by Waldram et al. (see for instance [6, 9]) to introduce the generalised geometry formalism also for RR fields and so spinors, *i.e.* the fermion sector of supergravity.
The structure group for the generalised tangent bundle $O(d,d)$ admits a double covering by the group $Spin(d,d)$.[2] Let us further assume that a structure $Spin(d) \times Spin(d)$ is allowed. We indicate as $S(C_\pm)$ the related spinor bundles to the sub-bundles $C_\pm$, with γ^a and $\gamma^{\bar{a}}$ the gamma matrices and, as anticipated above, with $\varsigma^\pm$ the sections of the spinor bundles, elements in $\Gamma(S(C_\pm))$ [34].
Owning this, and following again [6], we write the action of the generalised (spin) connection on $\varsigma^\pm$ as

$$D_M \varsigma^+ = \partial_M \varsigma^+ + \frac{1}{4} \Omega_M{}^{ab} \gamma_{ab} \varsigma^+ \ , \qquad D_M \varsigma^- = \partial_M \varsigma^- + \frac{1}{4} \Omega_M{}^{\bar{a}\bar{b}} \gamma_{\bar{a}\bar{b}} \varsigma^- \ . \tag{2.13}$$

[2] Actually the double cover is admitted by the connected component $SO(d,d)$, however, worrying about this is not relevant for the current discussion.

From these two derivatives we can build the following operators, uniquely determined, in the basis frames $\{\hat{E}^+_a\}$ and $\{\hat{E}^-_{\bar{a}}\}$,

$$D_{\bar{a}}\varsigma^+ = \left(\nabla_{\bar{a}} - \frac{1}{8}H_{\bar{a}bc}\gamma^{bc}\right)\varsigma^+ \ , \qquad \gamma^a D_a\varsigma^+ = \left(\gamma^a\nabla_a - \frac{1}{24}H_{abc}\gamma^{abc}\right)\varsigma^+ \ , \tag{2.14}$$

$$D_a\varsigma^- = \left(\nabla_a - \frac{1}{8}H_{a\bar{b}\bar{c}}\gamma^{\bar{b}\bar{c}}\right)\varsigma^- \ , \qquad \gamma^{\bar{a}} D_{\bar{a}}\varsigma^- = \left(\gamma^{\bar{a}}\nabla_{\bar{a}} - \frac{1}{24}H_{\bar{a}\bar{b}\bar{c}}\gamma^{\bar{a}\bar{b}\bar{c}}\right)\varsigma^- \ . \tag{2.15}$$

Note that no A appears, hence these quantities have no ambiguities, and are uniquely determined. This happens because of the elegant cancellation owing to the gamma matrices contractions with the A coefficients.
We can use this property to construct a generalised Ricci tensor. We will deal with these aspects in section 2.4.2, but firstly we need to introduce the generalised notion of curvature.

2.4 Generalised curvature

Once the concepts of generalised connection and torsion have been defined, it is natural to wonder if defining an analogous generalisation of Riemannian curvsature is possible. Let us recall how the curvature of a connection ∇ is defined in ordinary differential geometry before moving into the generalised case. The curvature notion is encoded by a $(1,3)$-tensor named *Riemann tensor* $\mathcal{R} \in \Gamma\left(\Lambda^2 T^*M \otimes TM \otimes T^*M\right)$ defined by the *curvature map*

$$\begin{aligned} \mathcal{R}\colon\ \Gamma(TM)\times\Gamma(TM)\times\Gamma(TM) &\longrightarrow \Gamma(TM) \\ (u,v,w) &\longmapsto \mathcal{R}(u,v,w) \end{aligned} \tag{2.16}$$

where

$$\mathcal{R}(u,v,w) := [\nabla_u, \nabla_v]\, w - \nabla_{[u,v]} w \ ,$$

and the Riemann tensor is defined from the curvature map (with respect to a coordinate basis) as

$$\begin{aligned} &\mathcal{R}_{\mu\nu}{}^{\alpha}{}_{\beta} := \langle e^\alpha, \mathcal{R}(e_\mu, e_\nu, e_\beta)\rangle \Longrightarrow \\ &\mathcal{R}_{\mu\nu}{}^{\alpha}{}_{\beta} v^\beta = [\nabla_\mu, \nabla_\nu]\, v^\alpha - T^\beta{}_{\mu\nu}\nabla_\beta v^\alpha \ , \end{aligned}$$

where in the first line we denoted with the angle brackets $\langle\cdot,\cdot\rangle$ the contraction between the one-form e^α and the $(0,3)$-tensor $\mathcal{R}(e_\mu, e_\nu, e_\beta)$.
Geometrically, the curvature – together with the torsion – encodes the dependence of the parallel transport from the path. This is related to what is commonly called *intrinsic curvature* of a space.

We want now to generalise these concepts to the case of generalised geometry. One might expect that simply defining an analogous "generalised Riemannian curvature map" is enough. Unfortunately, this is not the case, as we are going to see in what follows, since the "generalised Riemann tensor" will not be a tensor. However, we will show how is possible to restrict our interest to orthogonal subbundles to solve this problem.

2.4.1 Generalised curvature map

The most straightforward procedure to generalised the notion of curvature we can imagine is to define a map analogous to the curvature one in (2.16), where commutators, covariant derivatives and vectors are replaced by the respective corresponding objects in generalised geometry,

$$\begin{aligned} R\colon\ \Gamma(E)\times\Gamma(E)\times\Gamma(E) &\longrightarrow \Gamma(E) \\ (U,V,W) &\longmapsto R(U,V,W) \end{aligned}$$

where we have

$$R(U,V,W) := [D_U, D_V]\, W - D_{[\![U,V]\!]} W \ .$$

Now we would like to verify that the object we defined is indeed a tensor, *i.e.* $R(fU, gV, hW) = fghR(U,V,W)$, for all f, g, h scalar functions. However, what we found is that the map $R(U,V,W)$ does not define a tensor, in fact,

$$\begin{aligned} R(fU, gV, hW) &= [D_{fU}, D_{gV}]\, hW - D_{[\![fU,gV]\!]} hW = \\ &= fgh\big([D_U, D_V]\, W - D_{[\![U,V]\!]} W\big) - \frac{1}{2} h\langle U, V\rangle D_{(f\mathrm{d}g - g\mathrm{d}f)} W \ . \end{aligned}$$

We can see it is not linear respect to the first two arguments, and so it does not transform as a tensor. Although this, is still possible recover the notion of curvature by this procedure if we restrict our map to admit, as first two arguments only generalised vector fields that are orthogonal to each other, such that $\langle U, V\rangle = 0$. This makes $R(U,V,W)$ a good tensorial quantity [9, 39]. Consider, as example, $U \in C_+$ and $V \in C_-$, then we can express curvature generalised map in components with respect to a frame,

$$R(U,V,W) = u^a v^{\bar{a}} W^A R_{a\bar{a}}{}^B{}_A \hat{E}_B \ .$$

This quantity is a tensor and capital indeces A, B refer to the adjoint representation of the group $O(d) \times O(d)$.

As consequence of requiring R to be a tensor, we get that it cannot "mix" C_+ and C_- quantities, so $R_{a\bar{b}}{}^c{}_{\bar{d}} = R_{a\bar{b}}{}^{\bar{c}}{}_d = 0$. This because the connection D defining R is an $O(d) \times O(d)$-connection and so, once we fix $U \in C_+$ and $V \in C_-$, it cannot mix components of these subbundles.

It could seem that we achieved what we wanted, however the object we defined, also when restricted to orthogonal subbundles of E, is not unique. As anticipated above, we can use the fact that some contractions of gamma matrices with generalised connection give rise to uniquely defined object. This will be the content of next section.

2.4.2 Generalised Ricci tensor and scalar

Recall that in conventional differential geometry the *Ricci tensor* is defined by a contraction of the Riemann one,

$$\mathcal{R}_{\mu\nu} := \mathcal{R}_{\alpha\mu}{}^{\alpha}{}_{\nu} \ ,$$

such that the action of this tensor on a vector is given by the commutator of covariant derivatives,

$$\mathcal{R}_{\mu\nu} v^{\mu} = \left[\nabla_{\mu}, \nabla_{\nu}\right] v^{\mu} \ . \tag{2.17}$$

A further important quantity derived from the Riemann tensor is the so-called *Ricci scalar* given by

$$\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu} \ . \tag{2.18}$$

Once we have revisited all these useful concepts in differential geometry, we are ready to move into the generalised case.

We have seen above how is not possible, in general, defining a unique generalised Riemann tensor, however, it is possible define a unique generalised Ricci object and show it is a tensor. We anticipated that we would use the uniquely determined operators we found in (2.14). Thus, we write a generalise Ricci tensor as a quantity that acts in analogy with (2.17) [6], expressing everithing in the $\{\hat{E}^{\pm}\}$ frames we write

$$\begin{aligned} R^{+}_{a\bar{b}} v^{a}_{+} &:= \left[D_a, D_{\bar{b}}\right] v^{a}_{+} \ , \\ R^{-}_{\bar{a}b} v^{\bar{a}}_{-} &:= \left[D_{\bar{a}}, D_b\right] v^{\bar{a}}_{-} \ . \end{aligned} \tag{2.19}$$

These two objects, that at first sight seem different, actually are the same, since $R^{+}_{a\bar{b}} = -R^{-}_{\bar{b}a}$. Hence, we are allowed to define a unique generalised Ricci tensor $R_{a\bar{b}}$; we know it is a tensor from the properties of the generalised connection D, when we restrict, as above, to orthogonal subspaces as $C_{\pm}$.

Another (equivalent) possibility is to restrict to the orthogonal subspaces, where the generalised Riemann is actually a tensor, and then define two (a priori different) Ricci tensors by

$$\begin{cases} R^{+}_{\bar{b}b} := R^{+}_{a\bar{b}}{}^{a}{}_{b} \\ R^{-}_{a\bar{a}} := R^{-}_{a\bar{b}}{}^{\bar{b}}{}_{\bar{a}} \end{cases}$$

As shown, these are actually the same object (up to a sign) and equivalent to those defined in (2.19), acting separately on C_+ and C_-.
If we choose the two frames on $C_\pm$ to be aligned, that is $e^+_a = e^-_{\bar{a}}$, we can write the non-zero components of the generalised Ricci tensor as

$$R_{ab} = \mathcal{R}_{ab} - \frac{1}{4} H_{acd} H_b{}^{cd} + \frac{1}{2} \nabla^c H_{abc} \ , \tag{2.20}$$

in terms of the ordinary Ricci tensor $\mathcal{R}_{ab}$ and the flux 3-form H.
It is also worth to notice that we cannot define a generalised Ricci tensor by a simple contraction of the generalised Riemann, like $R_{AB} = R_{CA}{}^C{}_B$, but, as we have seen, we can still define a generalised Ricci tensor in the frame basis as

$$R_{AB} = \begin{pmatrix} R_{ab} & R_{a\bar{b}} \\ R_{\bar{a}b} & R_{\bar{a}\bar{b}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2d} R \delta_{ab} & R_{a\bar{b}} \\ R_{\bar{a}b} & \frac{1}{2d} R \delta_{\bar{a}\bar{b}} \end{pmatrix} \ ,$$

where R is the generalised Ricci scalar and the factor $1/2d$ is there for a normalisation that will become clear in a while.
Recall also the representation of the generalised metric G in the frame basis in (1.50), that, expressed in this fashion matrix form, becomes

$$G^{AB} = \begin{pmatrix} \delta^{ab} & 0 \\ 0 & \delta^{\bar{a}\bar{b}} \end{pmatrix} \ .$$

This allows us to write the generalised Ricci scalar in a way very close to (2.18) (the reason for the factor $1/2d$ is now evident),

$$R = G^{AB} R_{AB} \ ,$$

that, from the last expression, appears to be unique. Finally, we can express R in terms of $\mathcal{R}$ and H,

$$R = \mathcal{R} - \frac{1}{12} H^2 \ .$$

We are now ready to discuss what we anticipated at the end of chapter 1, when we introduced the generalised metric. We have in mind to construct the equation of motion of the NS-NS sector of type II supergravity in this generalised geometry formalism. This let us reformulate all field content of supergravity from a geometric point of view, similarly to Einstein's general relativity.

2.5 Supergravity Einstein-Hilbert action

Type II supergravity theories, which are the low energy limit of type II (IIA and IIB) superstring theories, are $(9+1)$-dimensional theories and have a bosonic, or NS-NS, sector made up by the massless fields $\{g_{\mu\nu}, B_{\mu\nu}, \phi\}$, where $\mu = 0, \ldots, 9$. With $g_{\mu\nu}$ we denote the *graviton*, a tensor corresponding to the 2nd-rank symmetric and traceless representation of the group $SO(8)$, the Wigner little group for massless states in 10 dimensions. $B_{\mu\nu}$ is the tensor transforming in the 2nd-rank antisymmetric representation of $SO(8)$ and the scalar field ϕ, a singlet for the group, is usually called the *dilaton*.
In this dissertation we have not treated the dilaton field, however, as in [9], it can be introduced in this formalism by taking into account an *extended generalised tangent bundle* $\tilde{E}$, given by $E \otimes \mathbb{R}^+$. As expected, relations involving the generalised metric, connection and curvature are slightly modified by the presence of the dilaton. For more details one can read for instance [6, 24, 34].
Let us concentrate on the fields $\{g_{\mu\nu}, B_{\mu\nu}\}$ for the moment. We can count how many components we have to determine, as in table 2.1, and so the maximum number of equations of motion we are allowed to impose.

Field	degrees of freedom
$g_{\mu\nu}$	$\frac{d(d+1)}{2}$
$B_{\mu\nu}$	$\frac{d(d-1)}{2}$

TABLE 2.1: Number of degrees of freedom carried by the fields. Note that we are considering fields off-shell in this counting.

From the table above, we can see that the total number of degrees of freedom carried by the dynamical variables $g_{\mu\nu}$ and $B_{\mu\nu}$ is d^2, exactly the same number of indipendent components of the generalised Ricci tensor $R_{a\bar{b}}$. This can suggest us an evocative form for the generalised action, analogous to the Einstein-Hilbert action we know (and love) in general relativity. For this reason we call

$$S_H = \frac{1}{2\kappa^2} \int R \tag{2.21}$$

the *generalised Einstein-Hilbert action*.
Varying S_H in order to obtain equations of motion we get the generalised version of the vacuum Einstein equations,

$$R_{a\bar{b}} = 0 \ .$$

They are d^2 equations, as required by the counting of degrees of freedom.
In fact, we can explicitly verify how the (2.21) reproduces the supergravity action for

the bosonic sector [9],

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^{10}x\sqrt{-g}e^{-2\phi}\left(\mathcal{R} + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{12}H^2\right) .$$

In order to obtain the supergravity action from the (2.21) we should fix $\phi \equiv 0$, since in our derivation we did not consider this field, as said above. Once we have eliminated the dilaton from our discussion, and setting $d = 10$, as the type II supergravity requires we have an evident equality from the two actions,

$$\begin{aligned} S_H &= \frac{1}{2\kappa^2}\int \mathrm{d}^{10}x|\mathrm{Vol}_G|R = \frac{1}{2\kappa^2}\int \mathrm{d}^{10}x\sqrt{-g}\ R = \\ &= \frac{1}{2\kappa^2}\int \mathrm{d}^{10}x\sqrt{-g}\left(\mathcal{R} - \frac{1}{12}H^2\right) , \end{aligned}$$

which is exactly the type II supergravity action without the dilaton. We have used the fact that $|\mathrm{Vol}_G|$ is the volume form associated to the generalised metric G in (1.44) and can be shown to be equal to $\sqrt{-g}$.

Just for completeness, we should say that when the dilaton is considered, even if the expression for the Ricci scalar is different, the form of the generalised Einstein-Hilbert action is always the (2.21).

To conclude, we should notice a subtlety, since we have just d^2 degrees of freedom, we are not allowed to contract the Ricci tensor to obtain a Ricci scalar. Indeed, in the (2.21) R must be intended as simply a shorthand notation for $G^{AB}R_{AB}$, and the equations of motion generated by the variation of S_B are the ones written above, since the generalised Ricci tensor fills all the available degrees of freedom with its components, as indicated in the table 2.1. In order to consider R as a proper object, encoding the dynamics of the fields, we should include the dilaton ϕ degree of freedom and so obtaining another equation of motion for it, $R = 0$, as showed in [6].

Furthermore, we could also include the RR sector in our analysis, as done in [9, 34] and as outlined above, by including the fermions field as source for the generalised curvature.

We have now achieved one of the main aims of this dissertation: reformulating type II supergravity theories in terms of generalised geometry, and geometrising their field content.

Finally, we would stress that is possible to formulate other kind of generalised geometries, see as example [25, 26], in which connection, metric and other structures can encode other physical properties, like equations of motion of 11-dimensional supergravity, coming from the M-theory [34, 35].

Chapter 3

Parallelisability and consistent truncations in Generalised Geometry

In this part of the dissertation we are going to cope with the notion of *parallelisability*. After a brief revision of the basic concepts in ordinary differential geometry we will move to generalised geometry, defining *general parallelisations* and *general Leibnitz parallelisations*, and exploring the consequences of their existence and how this is related to string theory.
The concept of parallelism is arising recently also in other applications in string theory, like in the study of perturbations, see for instance [40], but for what concerns this work these structures are interesting since they seems to be closely related to the existence of consistent truncations of higher dimensional theories on lower dimensional manifolds. For the 10-dimensional supergravity theory, this is the content of a conjecture formulated firstly by Waldram et al. in [41], but also appeared implicitly in [24]. We are not going to work on this conjecture, but we will find some examples of general Leibnitz parallelism and we will to undersand the consequences of the existence of this structure. At the end of the chapter, we will analyse an important result of this work. We will take into account a conjecture by Cvetic, Gibbons, Lu and Pope in [42] which states that taking a Lie group manifold G and reducing a theory on it, we are allowed to take a gauged theory in lower dimension with gauge group $G \times G$. We will show how this conjecture becomes natural and easily explicable in the framework of generalised geometry.

3.1 Parallelisable manifolds

In ordinary differential geometry a parallelisable manifold [3] is defined as follows,

Definition 3.1 (Parallelisation)**.** Given an n-dimensional differentiable manifold $\mathcal{M}$, a *parallelisation* – or an *absolute parallelism* – of $\mathcal{M}$ is a set $\{x_1, \ldots, x_n\}$ of n globally defined vector fields such that for all $p \in \mathcal{M}$, the set $\{x_1(p), \ldots, x_n(p)\}$ form a basis for the tangent space in p, $T_p\mathcal{M}$.
If $\mathcal{M}$ admits such a set, it is said to be *parallelisable*.

We can express the previous definition also saying that a manifold $\mathcal{M}$ is parallelisable if admits a *global frame*, since each basis vector $\hat{e}_a$ is a globally defined smooth vector field. This property of the manifold restricts the topology of the tangent bundle such that $T\mathcal{M}$ is a trivial bundle, *i.e.* if $\mathcal{M}$ is n-dimensional, then $T\mathcal{M} \cong \mathcal{M} \times \mathbb{R}^n$.
In fact, a parallelisation induces an isomorphism between tangent spaces in different points of the manifold. For example, aligning frames in each point, we can identify tangent spaces in that points. This assigns a connection with zero curvature to the manifold [43]. We can equivalently define the concept of parallelisation from the point of view of G-structures, infact, a parallelisation is a $\{e\}$-structure, where $\{e\}$ is the trivial group, containing only the identity element.
It is a well-known and remarkable result in algebraic topology – due to Bott and Milnor et al. [44, 45] – that the only parallelisable spheres are S^1, S^3, S^7, this is related to the existence of the normed division algebras $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$. Another famous result is the non-existence of a parallelisation for a 2-sphere. This statement descends directly from the so-called *hairy ball theorem*, a particular case of the *Poincaré-Hopf theorem* that is considered very important since it provides a link between topological properties of manifolds and analytical ones [46, 47].
In addition to the spheres we indicated above, there other examples of parallelisable manifolds. The simplest one is given by group maifolds. We can easily find a globally defined set of vector fields, forming a basis on T_gG in each point $g \in G$, *i.e.* a parallelisation. These are the *left(right)-invariant vector fields* $\hat{e}_a$ that are also related by the Lie algebra condition,

$$[\hat{e}_a, \hat{e}_b] = f_{ab}{}^c \hat{e}_c \ , \tag{3.1}$$

where $f_{ab}{}^c$ are coefficients that do not depend on the point of the manifold where we evaluate the Lie bracket of the fields[1] and hence are called *structure constants*. If such a parallelisation is admitted, it is possible to define a metric with constant coefficients (see for example [48]) $g = g_{ab}\, e^a \otimes e^b$, where g_{ab} are constants. This is nothing else than the *Cartan-Killing metric*, and with e^a we indicated the dual frame of $\hat{e}_a$.

[1]One could now compare with (1.28).

Note that among the parallelisable spheres, only $S^1 \cong U(1)$ and $S^3 \cong SU(2)$ are group manifolds, thus we can state immediately that the class of parallelisable manifolds is bigger than the class of Lie groups. In fact, there are some results in differential geometry stating that a *local group manifold i.e.* $\mathcal{M} \cong G/\Gamma$, where Γ is some discrete, freely-acting subgroup of the Lie group G, is a parallelisable manifold. A possible parallelisation is given precisely by the left (right) invariant vector fields of G if Γ acts on the left (right), moreover for any local group manifold, the parallelisation satisfies the (3.1).
Technically, this happens because the condition $M = G/\Gamma$ has some consequences on the left (right) invariant vector fields defined over the manifold. In particular, this set of fields, which plays the role of generators of the Lie algebra $\mathfrak{g}$, must satisfy the commutation relations (3.1), with the structure constants that do not depend on the point on M, as seen before.

3.1.1 Parallelisability in generalised geometry

We are now ready to generalise definition 3.1 to the generalised case.

Definition 3.2 (Generalised Parallelisation)**.** Given an n-dimensional manifold $\mathcal{M}$, a *generalised parallelisation* of $\mathcal{M}$ is a set of $2n$ globally defined generalised vector fields $\{X_A\}$ such that at any point $p \in \mathcal{M}$ the set $\{X_A\big|_p\}$ forms a basis of the fiber $E_p \cong T_p\mathcal{M} \oplus T_p^*\mathcal{M}$ of the generalised tangent bundle E. Furthermore, it is required the condition $< X_A, X_B >= delta_{AB}$.

It is immediate to verify that a globally defined generalised frame $\{\hat{E}_A\}$ is a generalised parallelisation. Topologically, the existence of a globally defined generalised frame, as in the ordinary case, means that the generalised tangent bundle is trivial, *i.e.* $E \cong \mathcal{M}^2 \times \mathbb{R}^{2n}$.
We can observe how the condition for a manifold to be parallelisable in the generalised sense are weaker than in the ordinary one. In fact, heuristically, we can think a section of E as the sum of a vector and a form. Then, even if a global (ordinary) frame may not exist, it could be possible to find two complementary sets of generalised vectors (equivalently two global generalised frames) defined such that the vector part does not vanish when the form part does and viceversa.

There exists a central point in our discussion: it is possible to define a generalised geometric analog of a local group manifold. This kind of space is a manifold M, endowed with a global generalised frame $\{\hat{E}_A\}$ such that

$$L_{\hat{E}_A}\hat{E}_B = F_{AB}{}^C\hat{E}_C \ , \tag{3.2}$$

where ${F_{AB}}^C$ are constants, in the sense they do not depend on the point on the manifold. By construction, E is trivial and frames in (3.2) defines a generalised parallelisation on the manifold M. This leads us to the definition of a special class of vector fields among parallelisations, the *generalised Leibnitz parallelisations* [41].

Definition 3.3 (Generalised Leibnitz Parallelisation)**.** A *generalised Leibnitz parallelisation* of an n-dimensional manifold $\mathcal{M}$ is a general parallelisation that in addition satisfies the further condition (3.2).

As seen before, we can define an associated spinor bundle to E. If the latter is trivial, also the former will be trivial as well and then, when equations for supergravity will be truncated (as in [35]) on a lower dimensional manifold, they gives a theory with the same number of supersymmetries as the higher dimensional one [41].
Note that the crucial point is the fact that the constants ${F_{AB}}^C$ are point-independent, in fact, as seen in (1.28), any frame can be expressed in the form of (3.2), where the F depend on the point.
As we will see in what follows, requiring a generalised Leibnitz parallelisation is more restrictive than requiring just a generalised one, but it is still less restrive than requiring the existence of an ordinary absolute parallelism, as defined in 3.1.

3.1.2 Consequences of the existence of a generalised parallelism

Following [24] we are going to prove that a necessary condition to admit a generalised Leibnitz parallelisation is not to be a local group manifold, but it is enough to be an homogeneous space, that is a coset space in the form G/H.
Firstly, we prove an useful lemma.

Lemma 3.4. *Dorfman derivative and Courant bracket coincide when they acts on a global section of the generalised frame bundle* $\{\hat{E}_A\}$.

Proof. This statement can be proved quite easily by looking at the definitions of the two objects, these imply, as seen in (1.7), that the difference between Courant bracket and Dorfman derivative is the exact differential of the inner product,

$$L_V W = [\![V, W]\!] + \mathrm{d}\langle V, W\rangle \ .$$

On the other hand, $\{\hat{E}_A\}$ is a global section of the global frame bundle (1.37), thus it holds

$$\eta(\hat{E}_A, \hat{E}_B) = \langle \hat{E}_A, \hat{E}_B\rangle = \eta_{AB} \ .$$

Hence,

$$L_{\hat{E}_A}\hat{E}_B = [\![\hat{E}_A, \hat{E}_B]\!] + \mathrm{d}\,(\eta_{AB}) = [\![\hat{E}_A, \hat{E}_B]\!] \ .$$

□

The previous lemma will be useful to prove a series of results in what follows.

Theorem 3.5 (Characterization of generalised Leibnitz parallelisable manifolds)**.** *Given a manifold $\mathcal{M}$, a necessary condition on $\mathcal{M}$ to admit a generalised Leibnitz parallelisation is to be expressible in the form $\mathcal{M} \cong G/H$,* i.e. *it must be an homogeneous space.*

Proof (sketch). Consider a generalised geometry on the base manifold $\mathcal{M}$.
Recall the properties **c1** and **c4** in the definition of Courant algebroid 1.2, and also recall that the difference between Courant bracket and Dorfman derivative is the inner product, as in (1.7). These implies the following relation on the Dorfman derivative,

$$\pi\,(L_V W) = \pi\big([\![V, W]\!] + \mathrm{d}\langle V, W\rangle\big) = \pi\big([\![V, W]\!]\big) = [v, w] = \mathcal{L}_v w \ ,$$

whose means that the Dorfman derivative reduces to the Lie one when projected on the tangent bundle $T\mathcal{M}$.
Now, consider a generalised Leibnitz parallelisation $\{\hat{E}_A\}$, defined as

$$\hat{E}_A = x_A - i_{x_A}B + \lambda_A \quad A = 1, \dots, 2d \ .$$

The (3.2) must be verified by the $\hat{E}_A$. In addition recall that, as we have seen in chapter 1, the Dorfman derivative defines a Leibnitz algebra.
Thus when projected by the anchor map, the Leibnitz algebra (3.2) reduces to a condition defining a Lie algebra

$$[x_A, x_B] = F_{AB}{}^C x_C \ . \tag{3.3}$$

We can see is also looking at the property **c2** of the Courant algebroid. In fact, also using the result of lemma 3.4 and the 3.2,

$$\begin{aligned}\mathrm{Jac}(\hat{E}_A, \hat{E}_B, \hat{E}_C) &= \frac{1}{3}\mathrm{d}\left(\langle[\![\hat{E}_A, \hat{E}_B]\!], \hat{E}_C\rangle + \textit{circ. perm.}\right) = \\ &= \frac{1}{3}\mathrm{d}\left(F_{AB}{}^D\eta_{CD} + \textit{circ. perm.}\right) = 0 \ ,\end{aligned}$$

where the last equality comes from the fact that the F are constants.
Hence, since we have a Jacobi identity, it defines a Lie algebra. Furthermore, from the previous relations [2] we have the following

$$\eta_{AB}F^B{}_{CD} + \eta_{BD}F^B{}_{CA} = 0 \ ,$$

which implies $F_{ABC} = F_{[ABC]}$.
Previous equation means that ${F_{AB}}^C =: {(T_A)_B}^C$ can be seen as the adjoint representation of the algebra with generators T_A and the latter is a Lie sub-algebra $\mathfrak{g}$ of $\mathfrak{o}(d,d)$ [24].
We can now state that the (3.3) is a realisation of the sub-algebra $\mathfrak{g}$ by the $2d$ vector fields x_A on the manifold. Since the manifold $\mathcal{M}$ is d-dimensional, eventually not all x_A can be linearly independent, and it is possible for some of them to be identically zero. Nonetheless, since the $\hat{E}_A$ are a basis of the generalised tangent bundle, there must exists a subset of x_A, *i.e.* at least d non-vanishing x, forming a basis of $T_p\mathcal{M}$ for all $p \in \mathcal{M}$.
We can now consider diffeomorphism generated by flows of these vector fields x, because of the construction above, we can argue there are d linearly independent flows, on a d-dimensional manifold, that means $\mathcal{M}$ is an homogeneous space, *i.e.* $\mathcal{M} \cong G/H$, where the group acting on it is G corresponding to the algebra $\mathfrak{g}$, generated by the d linearly independent combinations of x, and H is the group generated by the remaining vector fields.
For a complete and more rigorous proof, one can read [24]. □

An observation that will be useful in what follows is that we can think any sphere as an homeneous space: $S^d \cong SO(d+1)/SO(d)$. In this case, we have a natural action of $SO(d+1)$ on the manifold, defined by the Killing vector fields (given a metric on the sphere). While, $H \equiv SO(d)$ is the group that leaves points on the manifold invariant.

Previous results allow us to state also the fact that both Dorfman derivative and Courant bracket (that on $\hat{E}_A$ can be identified) define a Lie algebra. A very natural consequence, as we could expect, is that a manifold $\mathcal{M}$ which is parallelisable in the conventional sense is also Leibnitz parallelisable.

To conclude this section, we want to remark that if we restrict to consider global sections of the generalised frame bundle, we can forget about the difference between Courant bracket and Dorfman derivative, and moreover, we can treat both as Lie bracket. This fact has deep implications, and analysing them will be the aim of the rest of the chapter.

3.2 Consistent truncations and Leibnitz parallelism

We want now to give an explation about why we are interested in parallelisability in generalised geometry. For sure, it represents an interesting property from the mathematical point of view. It allows, for example, to understand better when differential structures on generalised bundles coincide, or it establishes a further link between generalised geometry and G-structures, etc. thus, it is a tool to better understand generalised geometry. However, we are interested in applications in physics, and more precisely, in string theory and supergravity. The physical interest in generalised parallelisations, and in particular in the Leibnitz ones, comes from the existence of a conjecture, like in [41], that links *consistent truncations* with the existence of generalised Leibnitz parallelisations.

Before exposing the conjecture, we are going to introduce consistent truncations and to explain their importance in string theory.

3.2.1 Dimensional reduction and consistent truncations

In order to give an idea of what consistent truncations are, we are going to introduce a *Kaluza-Klein dimensional reduction* [49] of a theory of a massless scalar field. Hystorically, it was proposed to incorporate the electromagnetism into the Einsten gravity, by adding a 4th spatial dimension [50, 51], and then operating a dimensional reduction, recover an effective field theory on the usual 4-dimensional manifold. At a later stage it had been recovered since the idea of compactifing extra-dimensions revealed to be very prolific in string theory. Now we indicate as Kaluza-Klein dimensional reduction simply the compactification on a circle.

Consider a theory of a massless scalar field φ on a d-dimensional ($d = p + 1$) manifold M. Moreover, suppose that one of the d dimensions (the pth for convenience) on M is periodic, *i.e* there is the identification $x^p \sim x^p + 2\pi R$. Hence, we can write

$$M \longrightarrow \mathcal{M} \times S^1_R \ ,$$

where R is the radius of the circle. We write coordinates on M as $x^I \equiv (x^\mu, y)$, where $I = 0, \ldots, p$, $\mu = 0, \ldots p - 1$ and $x^p \equiv y$.

The scalar field satisfies the d-dimensional Klein-Gordon equation

$$\Box\varphi = 0 \ ,$$

and, as usual, $\Box := \partial_I \partial^I$.

The periodicity of the p-th dimension allows us to write the Fourier series of the function along that direction

$$\varphi(x^\mu, y) = \sum_n \varphi_n(x^\mu) e^{-ip_n y} \ , \qquad p_n := \frac{n}{R} \ .$$

As a consequence of the periodicity condition, we obtained the quantisation of the momentum along the y-direction.
Inserting the Fourier series into the Klein-Gordon equation we get,

$$\partial_\mu \partial^\mu \varphi_n + \frac{n^2}{R^2} \varphi_n = 0 \qquad \forall n \in \mathbb{Z} \; .$$

We got an infinite tower of massive modes (for $n \neq 0$) with masses $m_n = n/R$. Hence, from a massless scalar field theory, we can recover a theory with infinite massive scalar fields, by dimensional reduction.
We are now ready to define consistent truncations. In the previous example, we supposed a free massless scalar field and this gave us an infinite tower of free massive scalar fields. If we introduce some interaction terms, we will obtain also interacting theories in lower dimensions. Then, a consistent truncation is a choice of a finite set of modes where the omitted ones are not sourced by the subset chosen. This is equivalent to say that the set of truncated modes has a dynamics wich is not affected by the other modes. This fact allows us to say that a solution to equations of motion in the lower dimensional theory, which is a linear combination of only truncated modes, remains a solution also on the higher dimensional manifold.
We can consider a simple example to understand what we mean for consistent truncations. Given a theory of two scalars with Lagrangian,

$$\mathcal{L} = \frac{1}{2} \left(\partial \varphi_1\right)^2 + \frac{1}{2} \left(\partial \varphi_2\right)^2 - \frac{m_1^2}{2} \varphi_1^2 - \frac{m_2^2}{2} \varphi_2^2 - g \varphi_1^2 \varphi_2 \; .$$

It generates the following equations of motion,

$$\begin{aligned} \partial_\mu \partial^\mu \varphi_1 + m_1^2 \varphi_1 &= -2g \varphi_1 \varphi_2 \; ; \\ \partial_\mu \partial^\mu \varphi_2 + m_2^2 \varphi_2 &= -g \varphi_1^2 \; . \end{aligned}$$

Therefore, we can observe how $\varphi_1 = 0$ is a consistent truncation, *i.e.* the evolution of φ_2 is given by a consistent (with the choice of suppresing φ_1) equation of motion, and fixed $\varphi_1 = 0$ at the initial time, it will remain fixed at all times. In other words, $\varphi_1 = 0$ is both a solution of the theory reduced to the only field φ_1, and of the full theory of the two scalar fields φ_1, φ_2. On the other hand, the dropping $\varphi_2 = 0$ is not consistent, since the dynamics of φ_1 will affect φ_2, due to the fact φ_1 acts as a source term for φ_2.
In our guide-example for dimensional reduction, we considered just the case of a scalar field, the situation can be slightly different in the case of a string, since a string can wrap around periodic dimensions with winding number $k \in \mathbb{Z}$, this property, apparently innocuous, gives origin to a famous and extremely important effect in string theory, the T-duality. We are not going to treat these topics in this dissertation, so for more details

one can read, for example [49, 52].

Consistent truncations are a very useful tools to study higher dimensional theories whose solutions would be very difficult to find otherwise. Nonetheless, only few examples of consistent truncations are known, in particular the so-called *Scherk-Schwarz reductions* [53, 54], defined in the case of a theory on $\mathcal{M} \times G$, where G is a group manifold. Furthermore it is a known result that parallelisable manifolds in the ordinary sense admit consistent truncations on them [42, 55].
The idea of Waldram et al. in [41] was to conjecture the existence of a consistent truncation everytime a general Leibnitz parallelism is admitted, since this represents the analogous of a group manifold, in the generalised background.
Here, we report the conjecture as stated in [41],

Conjecture. *Given a generalised Leibnitz parallelisation $\{\hat{E}_A\}$ on a generalised parallelisable manifold $\mathcal{M}$, there exists a consistent truncation on $\mathcal{M}$ preserving the same number of supersymmetries as the original theory.*

If the conjecture is proved true, it allows one to look as generalised Leibnitz parallelisations as a systematic way to find consistent truncations for supergravity and superstring theories.
In this thesis we are going to focus on studying generalised Leibnitz parallelisations, by constructing some examples.

3.3 Some generalised Leibnitz parallelisations

In [41] is showed that all spheres are parallelisable in the generalised sense, and further, they admitt a generalised Leibnitz parallelisation too. In general, it has been shown that one has to consider other kinds of generalised geometry, of the form $TM \oplus \Lambda^{d-2}T^*M$ to find a parallelisation for S^d.

3.3.1 A parallelisation for the 3-sphere

To introduce the method to find examples of generalised parallelisation we start by taking into account, as done in [41], a generalised parallelisation of S^3. In fact, for this manifold we can consider the generalised geometry we have studied so far, the $TM \oplus T^*M$ introduced by Hitchin and Gualtieri.
The first thing to notice is that $S^3 \cong SU(2)$ is a group manifold and so parallelisable

also in the ordinary sense. In order to construct a generalised globally defined frame we can choose left-invariant vector fields and their dual forms. Following the general method to construct general parallelisms given in [41] we embed S^3 (of radius R) into $\mathbb{R}^4$ by

$$\left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2 + \left(x^4\right)^2 \equiv \delta_{ij} x^i x^j = R^2 \ ,$$

with the Euclidean metric δ_{ij}, and $i = 1, \dots 4$. Then, the *constrained coordinates* $y^i := R^{-1} x^i$ can be introduced. In terms of y^i the 3-sphere embedding takes the form,

$$y^i y^j \delta_{ij} = 1 \ . \tag{3.4}$$

The last defines six Killing vector fields, the generators of $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. They can be represented in terms of y coordinates as $v_{ij} = R^{-1}\left(y_i \partial_j - y_j \partial_i\right)$.
Therefore, we can define the 3-form flux in a natural way on S^3, using the fact that it is an orientable 3-dimensional manifold, by the projection on S^3 of the $\mathbb{R}^4$ volume form. Firstly, for convenience also for later examples, we give a general formula for the volume form of a n-sphere as the projection of the $\mathbb{R}^{n+1}$ canonical volume form,

$$\mathrm{vol}\left(S^n\right) := \frac{R^n}{n!} \epsilon_{i_1 \cdots i_n j} y^j \mathrm{d}y^{i_1} \wedge \ldots \wedge \mathrm{d}y^{i_n} \ .$$

Hence, the flux form can be written as

$$H = \frac{2}{R} \mathrm{vol}\big|_{S^3} = \frac{R^2}{3} \epsilon_{ijkl}\ y^i \mathrm{d}y^j \wedge \mathrm{d}y^k \wedge \mathrm{d}y^l \ ,$$

which is combatible with the action of the v_{ij} Killing vectors in generalised geometry,

$$\mathcal{L}_{v_{ij}} g = 0 \ , \qquad \mathcal{L}_{v_{ij}} H = 0 \ .$$

Hence, we define our $O(3,3)$-generalised frame as

$$\hat{E}_{ij} := v_{ij} - i_{v_{ij}} B + * \left(R^2 \mathrm{d}y_i \wedge \mathrm{d}y_j\right) \ , \tag{3.5}$$

where $*$ denotes the Hodge star. Now, we want to prove that it is globally defined, *i.e.* it never vanishes. Note that v_{ij} vanishes when $y_i = y_j = 0$, but this does not make the whole $\hat{E}_{ij}$ vanish since the constraint (3.4) implies a relation also on the dual basis,

$$\delta_{ij} y^i \mathrm{d}y^j = y_i \mathrm{d}y^i = 0 \ ,$$

obtained by differentiating the (3.4). Thus the form must vanish when the corrispondent coordinate does not. In particular, the 2-form $\mathrm{d}y^i \wedge \mathrm{d}y^j$ vanishes on the equator $\left(y^i\right)^2 + \left(y^j\right)^2 = 1$, while v_{ij} is zero at the "poles", *i.e.* where $y^i = y^j = 0$.

We showed that (3.5) so defined never vanishes on S^3, while v_{ij} and $\mathrm{d}y^i \wedge \mathrm{d}y^j$ may vanish separately. This implies that S^3 admits the generalised parallelisation defined in (3.5).
Before proceding, it is worth to notice the role of the indeces (ij). We wrote $\hat{E}_{ij}$ by this pair of antisymmetric indeces running from 1 to 4, rather than the usual $A = 1, \ldots, 2d$ with $d = 3$. This to keep track of the natural definition of v_{ij} in these constrained coordinates and so of the $\mathfrak{so}(4)$ algebra. One can easy check that the number of independent components is actually the same (six) as should be.
Now, we prove that the parallelisation defined above is actually a generalised Leibnitz parallelisation. In fact, calculating the Dorfman derivative we get,

$$L_{\hat{E}_{ij}} \hat{E}_{kl} = [\![\hat{E}_{ij}, \hat{E}_{kl}]\!] = R^{-1} \left(\delta_{ik} \hat{E}_{lj} - \delta_{il} \hat{E}_{kj} - \delta_{jk} \hat{E}_{li} + \delta_{jl} \hat{E}_{ki} \right) \ , \tag{3.6}$$

which is a representation of the $\mathfrak{so}(4)$ algebra, as we stated above.
To recover the splitting of the $\mathfrak{so}(4)$ algebra into $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ we can define the *self-dual* and *antiself-dual* combinations of the $\hat{E}_{ij}$,

$$\hat{E}^{\pm}_{ij} := \hat{E}_{ij} \pm \frac{1}{2} \epsilon_{ijkl} \hat{E}_{kl} \ , \tag{3.7}$$

which the only non-zero components can be written as,

$$\begin{aligned} \hat{E}^{+}_{a} &= \frac{1}{2} \epsilon_{aij} \hat{E}^{+}_{ij} \\ \hat{E}^{-}_{\bar{a}} &= \frac{1}{2} \epsilon_{\bar{a}ij} \hat{E}^{-}_{ij} \ . \end{aligned}$$

In terms of these frame vectors we can write the algebra (3.6) as the direct sum of the two $\mathfrak{su}(2)$ algebras,

$$\begin{aligned} L_{\hat{E}^{+}_{a}} \hat{E}^{+}_{b} &= R^{-1} \epsilon_{abc} \hat{E}^{+}_{c} \ , \\ L_{\hat{E}^{-}_{\bar{a}}} \hat{E}^{-}_{\bar{b}} &= R^{-1} \epsilon_{\bar{a}\bar{b}\bar{c}} \hat{E}^{-}_{\bar{c}} \ , \\ L_{\hat{E}^{+}_{a}} \hat{E}^{-}_{\bar{b}} &= 0 \ . \end{aligned} \tag{3.8}$$

Finally, note that the decomposition of the $\mathfrak{so}(4)$ algebra,

$$\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3) \ ,$$

is related to the fact that the frame (3.5) defines a generalised metric on the 3-sphere, as seen in the section 1.6, since the inner product $\mathfrak{so}(4)$ restricted on the spaces $C_{\pm}$ defines two $\mathfrak{su}(2)$ inner products on these sub-bundles. This last observation will be followed in all the cases we are going to treat, and in particular, in section 3.3.4, it will be shown how is not possible to define a generalised metric on the generalised geometry over dS_3 and H^3.

From the point of view of generalised geometry, we have a splitting $SO(3,3) \longrightarrow SO(3)\times SO(3)$, where $SO(3,3)$ is the structure group acting on the generalised frame bundle. Accidentally, the algebra $\mathfrak{so}(3,3)$ is isomorphic to $\mathfrak{sl}(4)$ thus we can cosider $SL(4) \hookleftarrow SO(3)\times SO(3)$ therefore taking into account the $\hat{E}_{ij}$ properties of transformation under the separate $SO(3)$ factors leads us to the sub-frames in (3.7). In addition, note that ϵ_{ijkl} is the invariant tensor of $SL(4)$ and that it is related also to the structure constants of the $\mathfrak{so}(3)$ algebras.

3.3.2 A parallelisation for $S^2 \times S^1$

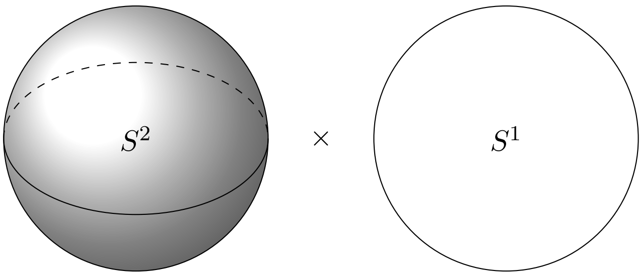

Figure 3.1

We now move to some different manifolds, in particular we are going to construct a generalised Leibnitz parallelism for $S^2 \times S^1$. It is still a 3-dimensional manifold, such that we can exploit our tools from the $TM \oplus T^*M$ generalised geometry, but it is a product manifold and a bit more interesting than the previous S^3 case, due to the different nature of the frames on the two factors that we use in order to construct a generalised frame.

We know from [41] that both S^2 and S^1 are general Leibnitz parallelisable. A result from the ordinary differential geometry states that a product of manifolds with at least one of them is parallelisable is parallelisable too. Although an analogous theorem has not yet been proved in generalised geometry, we are interested in a more restricitve property than the parallelisability, we are looking at the Leibnitz parallelisability condition, thus we can adfirm that $S^2 \times S^1$ has a general Leibnitz parallelism and we are going to build it explicitly.

We could embed the manifold into $\mathbb{R}^5 \cong \mathbb{R}^3 \times \mathbb{R}^2$, with the usual euclidean metric on each factor, but what we actually do is embedding just the S^2 manifold into $\mathbb{R}^3$ and using the constrained coordinates y^i as before on it, with $i = 1 \ldots 3$. On S^1, instead, we choose an atlas such that we are allowed to consider $\psi \in (0, 2\pi)$ as the only coordinate on the circle. The first issue arises when we want to define the 3-form flux H. Again, we make the ansatz of choosing H proportional to the volume form on the manifold. On

S^2 and S^1 we have, respectively,

$$\text{vol}_2 = -\frac{1}{2}\epsilon_{ijk}y^i \mathrm{d}y^j \wedge \mathrm{d}y^k \ , \qquad \text{vol}_1 = \mathrm{d}\psi \ ,$$

where once we denoted as ψ the coordinate on S^1, the volume form is just the dual basis in that coordinates, $\mathrm{d}\psi$. Hence, the flux form is written as

$$H = -\frac{1}{2}\epsilon_{ijk}y^i \mathrm{d}y^j \wedge \mathrm{d}y^k \wedge \mathrm{d}\psi \ . \tag{3.9}$$

Thus, we can define the generalised frame on this manifold by the following two sets of generalised vectors,

$$\hat{E}_A = \begin{cases} \hat{E}_i & = v_i - i_{v_i}B + y_i \mathrm{d}\psi \\ \hat{E}'_i & = y_i\partial_\psi - i_{y_i\partial_\psi}B + \epsilon_{ijk}y^j \mathrm{d}y^k \end{cases} \qquad A = 1, \ldots, 6 \ . \tag{3.10}$$

In the previous $v_i := 1/2\,\epsilon_{ijk}y_j\partial_k$ are the $SO(3)$ Killing vector fields, generating the isometries on the sphere.
To verify the generalised frame defined by (3.10) never vanishes, as in the previous case, we can observe the following.
Consider the vector part of $\hat{E}_i$, v_i. It vanishes, by definition, when $y_j = y_k = 0$, this constrains $y_i = 1$, that corresponds to a pole of the sphere on the y_i axis, and there the form part $y_i\mathrm{d}\psi$ is non-zero, since $\mathrm{d}\psi$ is constant and never vanishes on S^1. Thus $\hat{E}_i$ is globally defined. For $\hat{E}'_i$, the vector part vanishes when y_i does, but when this happens $y^j\mathrm{d}y^k$ must be non-zero, from the relation defining the 2-sphere $y^iy^i = 1$.

In analogy with the S^3 case, we can calculate the Dorfman derivatives of frame elements to check if they represents a generalised Leibnitz parallelisation. Indeed, we find that,

$$\begin{aligned} L_{\hat{E}_i}\hat{E}_j &= [\![\hat{E}_i, \hat{E}_j]\!] = -\epsilon_{ijk}\hat{E}_k \ ; \\ L_{\hat{E}_i}\hat{E}'_j &= [\![\hat{E}_i, \hat{E}'_j]\!] = -\epsilon_{ijk}\hat{E}'_k \ ; \\ L_{\hat{E}'_i}\hat{E}'_j &= [\![\hat{E}'_i, \hat{E}'_j]\!] = 0 \ . \end{aligned}$$

One can find the detailed calculation in the appendix A. Here, we want to focus on the algebra the relations above define, indeed it can be shown that they provide a representation for the $\mathfrak{iso}(3)$ algebra, corresponding to the group of isometries of the 3-dimensional Euclidean space $ISO(3) \cong SO(3) \ltimes \mathbb{R}^3$.
To conclude, we can verify the orthonormality condition of the $O(d,d)$ frame (3.10),

$$\eta\left(\hat{E}_i, \hat{E}_j\right) = 0 \ , \qquad \eta\left(\hat{E}'_i, \hat{E}'_j\right) = 0 \ , \qquad \eta\left(\hat{E}_i, \hat{E}'_j\right) = \frac{1}{2}\delta_{ij} \ .$$

These define a metric as (1.2), the explicit proof is given in the appendix A. The diagonalisation of the metric, and then the splitting $O(3,3) \longrightarrow O(3) \times O(3)$, is made by choosing the following linear combination of the frames,

$$\hat{E}_i^{\pm} := \hat{E}_i \pm \hat{E}'_i \ .$$

3.3.3 An higher dimensional example: $S^3 \times S^3$

We can also consider the higher dimensional case $S^3 \times S^3$. It is interesting since, instead of considering a generalised geometry $TM \oplus \Lambda^4 T^* M$ as [41] may suggest, we keep on with our choice of the canonical $TM \oplus T^* M$, showing that constructing a generalised global frame is possible also in this framework.
Firstly, let us define the volume form on this manifold, embedding our manifold into $\mathbb{R}^8 \cong \mathbb{R}^4 \times \mathbb{R}^4$,

$$\begin{aligned}
\mathrm{vol} &:= \mathrm{vol}_3 \wedge \mathrm{vol}_3 = \\
&= \left(\frac{R^3}{3!} \epsilon_{i_1 \cdots i_4} y^{i_1} \mathrm{d}y^{i_2} \wedge \ldots \wedge \mathrm{d}y^{i_4} \right) \wedge \left(\frac{R^3}{3!} \epsilon_{i_1 \cdots i_4} t^{i_1} \mathrm{d}t^{i_2} \wedge \ldots \wedge \mathrm{d}t^{i_4} \right) = \\
&= \Lambda^2 \left(\mathrm{vol}_3 \right) \ ,
\end{aligned}$$

where we used y and t costrained coordinates on the two spheres.
We are now going to construct the generalised global frame. Recall that with the base manifold $S^3 \times S^3$ the generalised tangent bundle $T\left(S^3 \times S^3\right) \oplus T^*\left(S^3 \times S^3\right)$ admits a structure group $SO(6,6)$ that can split into $SO(6) \times SO(6)$. Let now take into account the algebras of these groups. For $\mathfrak{so}(6)$ there exists the isomorphism of algebras,

$$\mathfrak{so}(6) \cong \mathfrak{su}(4) \supset \mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3) \ .$$

We have taken the sub algebra $\mathfrak{so}(4)$ of $\mathfrak{so}(6)$ since we want to consider $SO(6,6) \supset SO(6) \times SO(6) \supset SO(3) \times SO(3)$, in analogy with the S^3 case.
Thus, focusing on one $\mathfrak{so}(3) \cong \mathfrak{su}(2)$ piece, we can define separately on the two spheres,

$$\begin{cases} \hat{E}_{ij}^{(1)} = v_{ij}^{(1)} - i_{v_{ij}^{(1)}} B + * \left(R^2 \mathrm{d}y^i \wedge \mathrm{d}y^j \right) \\ \hat{E}_{ij}^{(2)} = v_{ij}^{(2)} - i_{v_{ij}^{(2)}} B + * \left(R^2 \mathrm{d}t^i \wedge \mathrm{d}t^j \right) \end{cases} \tag{3.11}$$

where $v_{ij}^{(1)}$ and $v_{ij}^{(2)}$ are the Killing vectors on the two spheres.
Then, the 4-indeces object

$$\hat{E}_{ijIJ} := \left\{ \hat{E}_{ij}^{(1)}, \hat{E}_{ij}^{(2)} \right\}$$

provides a globally defined generalised frame, that is, a generalised parallelisation. Moreover, it satisfies also the $\mathfrak{so}(4) \oplus \mathfrak{so}(4)$ algebra relations,

$$L_{\hat{E}^{\alpha}_{ij}} \hat{E}^{\beta}_{kl} = [\![\hat{E}^{\alpha}_{ij}, \hat{E}^{\beta}_{kl}]\!] = \delta^{\alpha\beta} R^{-1} \left(\delta_{ik} \hat{E}^{\alpha}_{lj} - \delta_{il} \hat{E}^{\alpha}_{kj} - \delta_{jk} \hat{E}^{\alpha}_{li} + \delta_{jl} \hat{E}^{\alpha}_{ki} \right) . \tag{3.12}$$

Where there is no sum on $\alpha = 1, 2$.

To be very general, we could consider different radii for the spheres. The construction is in fact the same, the only difference is that an index α will appear also on the radius in the (3.12).

To conclude, we can also define two couples of self-dual and anti self-dual vectors

$$\left(\hat{E}^{\alpha}_{ij} \right)^{\pm} := \hat{E}^{\alpha}_{ij} \pm \frac{1}{2} \epsilon_{ijkl} \hat{E}^{\alpha}_{kl}$$

that makes the $\mathfrak{so}(4)$ split into two $\mathfrak{su}(2) \cong \mathfrak{so}(3)$ algebras, and so define a generalised metric.

The proof of these facts descends directly from what we have seen in the construction of a Leibnitz parallelism for S^3, and consequently, the (3.11) provides a Leibnitz parallelism for $S^3 \times S^3$.

3.3.4 Homogeneous spaces

We now want to look at some more exotic 3-dimensional spaces, H^3, dS_3, AdS_3. All of them are maximally symmetric and writable as coset spaces, thus we can generalise the procedure we have seen for S^3 to these spaces, construct some globally defined generalised frames and verify that they satisfy the Leibnitz parallelisability condition (3.2). These manifolds appear in several applications and, for various reasons, they are largely studied in both mathematics and physics.

We can present them in a table,

Hyperbolic Space	$H^3 \cong \frac{SO(1,3)}{SO(2,1)}$
de Sitter Space	$dS_3 \cong \frac{SO(3,1)}{SO(2,1)}$
Anti-de Sitter Space	$AdS_3 \cong \frac{SO(2,2)}{SO(2,1)}$

The key idea to study these manifolds is to embed them in the same space $\mathbb{R}^4$, endowed in each case with a metric $\eta^{(p,q)}_{ij}$ with signature (p, q) – meaning that p eigenvalues are positive, q negatives – that may vary in each situation. In this way, the embedding expressions will have the same form, written in terms of the metric $\eta^{(p,q)}_{ij} x^i x^j = R^2$, the choice of the metric on $\mathbb{R}^4$, the embedding space, will lead to consider the different manifolds with a single approach. Thus, we can show the explicit choices of the metric to describe the various spaces,

Space	Embedding	Signature (p,q)
S^3	$x_1^2+x_2^2+x_3^2+x_4^2=R^2$	$(4,0)$
dS_3	$x_1^2+x_2^2+x_3^2-x_4^2=R^2$	$(3,1)$
AdS_3	$x_1^2+x_2^2-x_3^2-x_4^2=R^2$	$(2,2)$
H^3	$x_1^2+x_2^2+x_3^2-x_4^2=-R^2$	$(1,3)$

TABLE 3.1: Different choices of $\eta_{ij}^{(p,q)}$ for the various spaces.

As example, let us consider the dS_3 manifold. We want to take advantage from the costruction in 3.3.1 for S^3 to write a generalised frame and, by calculating Dorfman derivatives, showing that it satisfyes to a Lie algebra.
Geometrically dS_3 can be seen as the quotient group $SO(3,1)/SO(2,1)$, thus following our previous example, we construct the globally defined frame as

$$\hat{E}_{ij} = M_{ij} - i_{M_{ij}} B + * \left(R^2 \mathrm{d}y_i \wedge \mathrm{d}y_j\right) \ . \tag{3.13}$$

We indicated with $M_{ij} := 1/2\, \epsilon_{ijkl}\, y_k \partial_l$ the generators of $SO(3,1)$ whose form should be familiar in physics, since they are the generators of the Lorentz algebra, *i.e.* the 4-angular momentum tensors. One can verify the frame (3.13) is globally defined – the argument follows quite closely the one given for S^3.
We can so calculate Dorfman derivatives (or Courant brackets) between the frame (3.13) elements, obtaining an algebra formally analogous to (3.6), but slightly different because of the different choice of the metric $\eta_{ij}^{(p,q)}$ counting for the difference in the embedding expression in 3.1. The algebra has a nice expression as follows

$$L_{\hat{E}_{ij}} \hat{E}_{kl} = R^{-1} (\eta_{ik}^{(p,q)} \hat{E}_{lj} - \eta_{il}^{(p,q)} \hat{E}_{kj} - \eta_{jk}^{(p,q)} \hat{E}_{li} + \eta_{jl}^{(p,q)} \hat{E}_{ki}) \ , \tag{3.14}$$

where $(p,q) \equiv (3,1)$ in this case. The (3.14) provides actually the expression for the algebras of all other cases indicated in 3.1, with an opportune choice of the signature (p,q).

We can also study the possibility of putting a generalised metric structures on these spaces. In fact, not all of them admit a generalised metric since the algebras defining them do not all admit a splitting.
As before, we should find a splitting, by taking what we called "self-dual" and "antiself-dual" combinations of the frame elements. The only algebra for which this procedure is possible is the $\mathfrak{so}(2,2)$ defining the anti-de Sitter space. It splits into

$$\mathfrak{so}(2,2) \cong \mathfrak{so}(2,1) \oplus \mathfrak{so}(2,1) \ . \tag{3.15}$$

This fact implies that we can find a generalised metric G by looking at a combination of our frame elements forming a quantity transforming in the vector representation of the two sub-algebras.
However, in the other cases, the 3-hyperboloid H^3 and the de Sitter space dS_3, for the algebras $\mathfrak{so}(3,1)$ and $\mathfrak{so}(1,3)$ a natural split as the (3.15) does not exist. Although this, one can consider complexified algebras and studying their relations by the chain of isomorphisms

$$\mathfrak{so}(3,1)_{\mathbb{C}} \cong \mathfrak{so}(1,3)_{\mathbb{C}} \cong \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}} \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \ .$$

The last two of these define a splitting by the (complex) linear combination

$$\hat{E}^{\pm}_{ij} := \hat{E}_{ij} \pm \frac{i}{2}\epsilon_{ijkl}\hat{E}_{kl} \ .$$

This does not define a generalised metric G as in 1.3, but a slightly different object that still encodes g and B fields. This object is a generalised tensor J defining a *generalised complex structure* [17, 19]. These kinds of structures are very interesting from a mathematical point of view, indeed these are structure originally studied by Gualtieri and Hitchin in their works.
We would believe that the emerging of these kind of more general structures could give a possible suggestion to extend the approach of generalised geometry to the study of physical theories with a description in terms of these generalised complex structures J, which could be thought as the generalised analogous of the symplectic structure. Or more simply, it could indicate a deeper connection between theoretical physics and mathematics that would need further investigations, to better understand both subjects.

To conclude, we may suggest as a further development of these procedures, an analogous generalisation of the construction of general Leibnitz parallelisations for spheres S^d in [41] to the corresponding homegeneous spaces H^d, AdS_d and dS_d like the ones in the table 3.1, but in d dimensions, using the generalised geometries $TM \oplus \Lambda^{d-2}T^*M$ as for the spheres.

3.4 Dimensional reduction and gauge groups

In this section we are going to present an important result of this work. We give a proof of a theorem relating to the dimensional reduction of a supergravity theory on a manifold M, reduced on another (lower dimensional) manifold $\mathcal{M}$, compactifying on a (local) group manifold G.

This result is an example of how the generalised geometry approach, developed by Hitchin, Gualtieri et al. in mathematics, and then brought to physics by the work of Hull, Waldram, Petrini et al., is not just an elegant reformulation of the supergravity theories and a geometrisation of some field theories we already knew, but it establishes an useful way to achieve new knowledge about theoretical physics.
Hence, we state the following theorem, firstly considered in [42],

Theorem 3.6. *Given a supergravity theory in* 10 *dimensions, on a manifold* $\mathbb{M}_{10}$*, after dimensional reduction on a (local) group manifold* G*, the theory in the reduced-dimensional manifold* $\mathcal{M}$ *admits a gauge group* $G \times G$*.*

This result may appear surprising, due to the presence of the double product of the group G. We will see, how this arises quite naturally once we make use of generalised geometry tools. The two factors correspond to the right and left invariant vector fields forming the algebras of G, and so sometimes the product group may be denoted by $G_L \times G_R$.
This comes from the fact that a consistent truncation is always possible on a group manifold. Moreover any G can be seen as the coset $(G \times G)/G$ and this is exactly the condition to admit a general Leibnitz parallelism, related to the algebra $\mathfrak{g} \oplus \mathfrak{g}$. For these reasons, we have the hint to analyse the topic making use of our generalised geometric tools.

A further analysis of the theorem

We are going to prove the theorem 3.6 following a construction that leads us to write down explicitly the algebra relations for the Lie algebra of the group $G \times G$.
Consider a supergravity theory on $\mathbb{M}_{10}$ and perform a dimensional reduction on a group manifold G,

$$\mathbb{M}_{10} \longrightarrow \mathcal{M} \times G$$

Cosider the base manifold G, and a generalised geometry defined on it by the short exact sequence

$$0 \longrightarrow T^*G \xrightarrow{\iota} E \xrightarrow{\pi} TG \longrightarrow 0\ ,$$

as seen in chapter 1.
In addition, take into account the generalised frame that provides a *General Leibnitz Parallelisation*, $\{E^+_a\}$ and $\{E^-_{\bar{a}}\}$, defined from the *left* and *right-invariant vector fields* of the group G, as follows

$$\begin{cases} E^+_a = r_a + \rho_a - i_{r_a} B \\ E^-_{\bar{a}} = l_{\bar{a}} - \lambda_{\bar{a}} - i_{l_{\bar{a}}} B \end{cases} \tag{3.16}$$

Where the forms ρ^a and $l_{\bar{a}}$ are defined as the dual basis with respect to r_a and $l_{\bar{a}}$, *left* and *right-invariant vector fields*. In other words

$$i_{r_a}\rho^b = \delta_a{}^b, \qquad i_{l_{\bar{a}}}\lambda^{\bar{b}} = \delta_{\bar{a}}{}^{\bar{b}}, \tag{3.17}$$

and we use the *Cartan-Killing form* to raise and lower indeces,

$$g = g_{ab}\,\rho^a \otimes \rho^b = g_{\bar{a}\bar{b}}\,\lambda^{\bar{a}} \otimes \lambda^{\bar{b}}. \tag{3.18}$$

Recall the General Liebenitz Parallelisation conditions are

$$L_{E_a^+}E_b^+ = F_{ab}{}^c E_c^+ \tag{3.19a}$$
$$L_{E_{\bar{a}}^-}E_{\bar{b}}^- = F_{\bar{a}\bar{b}}{}^{\bar{c}} E_{\bar{c}}^- \tag{3.19b}$$
$$L_{E_{\bar{a}}^-}E_b^+ = 0 \tag{3.19c}$$

One can now impose (3.19) in order to find some conditions on the B field, or better its field strength $H = \mathrm{d}B$, defined globally.
For convenience one can points out Lie derivatives of quantites that are involved in later calculations,

$$\begin{aligned} \mathcal{L}_{r_a} r_b &= F_{ab}{}^c r_c \qquad &&\textbf{algebra transformation} \\ \mathcal{L}_{r_a} \rho^b &= F_{abc}\rho^c \qquad &&\textbf{adjoint transformation} \end{aligned}$$

Now, consider (3.19a) first and recall definition of Dorfman derivative (1.5),

$$\begin{aligned} L_{E_a^+}E_b^+ &= [r_a, r_b] + \mathcal{L}_{r_a}\rho^b - \mathcal{L}_{r_a} i_{r_b} B - i_{r_b}\mathrm{d}\rho^a + i_{r_b}\mathcal{L}_{r_a}B + i_{r_a} i_{r_b} H = \\ &= F_{ab}{}^c r_c + \mathcal{L}_{r_a}\rho^b - i_{[r_a, r_b]}B - i_{r_b}\mathrm{d}\rho^a + i_{r_a} i_{r_b} H = \\ &= F_{ab}{}^c r_c + F_{abc}\rho^c - F_{abc} i_{r_c} B - i_{r_b}\left(\mathrm{d}\rho^a + i_{r_a} H\right) \end{aligned}$$

This is equal to $F_{ab}{}^c E_c^+$ if and only if

$$i_{r_a} H = F^a_{\ bc}\,\rho^b \wedge \rho^c \tag{3.20}$$

since in this basis $\mathrm{d}\rho^a = -F^a_{\ bc}\,\rho^b \wedge \rho^c$. We can prove the last statement by a simple application of definition of exterior derivative of a 1-form. Consider a 1-form ω, then $\mathrm{d}\omega$ is a 2-form and acts on two vector fields to give a scalar function,

$$\mathrm{d}\omega\,(X, Y) = X\,[\omega\,(Y)] - Y\,[\omega\,(X)] - \omega\big([X, Y]\big).$$

Now, applying the last expression to the form ρ^a and vector fields r_b and r_c, also recalling that $\rho^a(r_b) = \delta^a{}_b$ we find

$$\mathrm{d}\rho^a\left(r_b, r_c\right) = r_b\left[\delta^a{}_c\right] - r_c\left[\delta^a{}_b\right] - \rho^a\big(\left[r_b, r_c\right]\big)$$

after substituting the algebra relation for r_a. Then, using linearity of the actions and the fact that a vector acting on a constant function gives zero, we get

$$\mathrm{d}\rho^a\left(r_b, r_c\right) = -F^a_{\ bc} \tag{3.21}$$

this implies

$$\mathrm{d}\rho^a = -F^a_{\ bc}\,\rho^b \wedge \rho^c. \tag{3.22}$$

With an analogous calculation we get a condition for H in the basis of forms $\{\lambda^{\bar{a}}\}$. This condition can be written as

$$i_{l_{\bar{a}}} H = -F_{\bar{a}\bar{b}\bar{c}}\,\lambda^{\bar{b}} \wedge \lambda^{\bar{c}} \tag{3.23}$$

Recostructing the 3-form we get

$$H = \frac{1}{3!} F_{abc}\,\rho^a \wedge \rho^b \wedge \rho^c = -\frac{1}{3!} F_{\bar{a}\bar{b}\bar{c}}\,\lambda^{\bar{a}} \wedge \lambda^{\bar{b}} \wedge \lambda^{\bar{c}}. \tag{3.24}$$

The minus sign is there since the different sign of the form part between left-invariant and right-invariant generalised frames (3.16).

The reason to define frames with this sign is that we want frames to be compatibles with the $O(d) \times O(d)$-structure defined by the generalised metric. Moreover, we can prove that the mutual orthogonality condition between the "plus" and the "minus" frames,

$$\langle E^+_a, E^-_{\bar{a}} \rangle = 0 \tag{3.25}$$

comes from only from the nature of right and left-invariant vector fields and the Cartan-Killing metric they induce. This is an issue of consistency, infact we have two relations (the $O(d)$ conditions on the two frames) and two dregrees of freedom (the actions of basis vectors on forms, we set this in (3.17)), and a further relation like (3.25) cannot be independent.

From direct calculations indeed, we can prove the condition (3.25) is equivalent to

$$i_{r_a}\lambda_{\bar{a}} = i_{l_{\bar{a}}}\rho_a \tag{3.26}$$

that, in turn, can be proved as follows: consider the Cartan-Killing metric form (3.18) acting on the two vectors r_c and $l_{\bar{c}}$, in other words take into account the contraction

$i_{r_c} i_{l_{\bar{c}}} g$,

$$i_{r_c} i_{l_{\bar{c}}} g = g_{ab} \delta^a_c i_{l_{\bar{c}}} \rho^b = g_{\bar{a}\bar{b}} \delta^{\bar{a}}_{\bar{c}} i_{r_c} \lambda^{\bar{b}} =$$
$$= i_{r_c} \lambda_{\bar{c}} = i_{l_{\bar{c}}} \rho_c,$$

where we have used the property of raising and lowering indeces by the metric contraction.
The last line is exactly the condition (3.26), and this also proves the (3.25).

To sum up, we have proved the consistency of relations defining the algebras with the fact that the frame constructed by left and right invariant vector fields is a general Leibnitz parallelisation, and so it could provide consistent truncations according to the conjecture expressed above.
Moreover, we have constructed a generalised frame globally defined whose elements form the algebra $\mathfrak{g}_R \oplus \mathfrak{g}_L$ by relations (3.19), here rewritten,

$$L_{E_a^+} E_b^+ = F_{ab}{}^c E_c^+ \qquad L_{E_{\bar{a}}^-} E_{\bar{b}}^- = F_{\bar{a}\bar{b}}^{\bar{c}} E_{\bar{c}}^- \ . \tag{3.27}$$

Hence, we can state the group related to this algebra is $G_L \times G_R$ and therefore the manifold admits a general Leibnitz parallelism. This, according to the conjecture we stated above, may prove that the theorem 3.6 provides a way to relate consistent truncations in the frame of generalised geometry, and the gauge group of reduced theories.

As example of the theorem 3.6, we have already considered in section 3.3.1 the case of a general parallelisation S^3. We found that the group associated to the Lie algebra of the frames was $SO(4)$ which is actually isomorphic to a product of the same group, $SU(2) \times SU(2)$. In fact, we defined the self-dual combinations of frames $\hat{E}^{\pm}$ in (3.7) and these are precisely the left and right invariant vector fields under $SU(2) \cong SO(3)$, as can be noted explicitly by the presence of the invariant tensor ϵ.
As showed in (3.8), we can see that in this case we have two separated algebras, satisfying independent conditions, as stated in (3.27).

Conclusions

The first part of this thesis is focused on the concepts of generalised geometry. It is a development of the tools we have used in the second part. We concentrated our attention, following the recent literature, on the construction of the generalised tangent bundle, defined as the extension of the tangent bundle by its dual, the cotangent bundle. This, as seen, produces the arising of a new kind of structure. The $O(d,d)$ group is the structure group of the generalised bundle, and plays an analogous role of the $GL(d)$ group in ordinary differential geometry. As consequence, we have a natural metric arising, defining an $O(d,d)$ inner product on the bundle. We gave definitions of the Dorfman derivative and the Courant bracket, to encode both diffeomorphism and gauge symmetries. After the introduction of the generalised metric as a splitting of bundles, we stated that the structure group also splits into $O(d) \times O(d)$ and we showed how these objects encodes completely the degrees of freedom of the two transformations g and B, representing the fields in the supergravity bosonic spectrum. We went on in our study of the generalised geometry, constructing the generalised connection and, after the definition of the generalised torsion, we stated the necessary conditions to have a generalised analogous of the Levi-Civita connection, *i.e.* a torsion free, metric compatible connection structure. After this, the natural prosecution was the construction of the Riemannian tensor and the analysis of the notion of curvature in this context. We found that the generalised analogous of the Riemannian curvature map fails to be a tensor. Although this, it is still possible to define a unique generalised Ricci tensor and a well-defined (for our purposes) notion of curvature. This led us to geometrise the fields g and B, in the sense that we wrote their dynamics by an action composed by only geometrical objects, *i.e.* the generalised metric and the generalised Ricci tensor. Then, we showed how from the action we wrote is possible to recover the type II supergravity action for the NS-NS sector.

Completed this first part of revision, we focused on the concept of parallelisability, in particular on a more restrictive structure, the analogous of a local group manifold in generalised geometry. This object are characterised by what is called a general Leibnitz parallelisation condition, that is a condition on the generalised frame bundle. In the

thesis we gave some examples and we found a remarkable fact, when we restrict this kind of parallelisations to the Generalised Frame Bundle all the differential structures we developed (*i.e.* Dorfman derivative, Courant Bracket) reduce to a Lie Algebra structure. This allows us to classify the spaces we studied by the correspond Lie algebras. By this hint we generalised the work of Waldram in [41] to some other examples, like $S^3 \times S^3$ or $S^2 \times S^1$ and the homegeneous spaces.
As seen in the main text, these arguments are closely related to the existence of consistent truncations on manifolds in string and supergravity theories, and if the conjeture we stated was actually true, this would lead to a reduction of a difficult geometrical problem to a classification problem in the theory of Lie algebras.
Our examples show how is possible to find quite a large class of spaces admitting a general Leibnitz parallelisation, and so with a coset space structure, the analogous of local group manifold in this context.

Despite these results, many questions remain unaswered and many others arise. First of all, a proof of the conjecture relating Leibnitz parallelisms and consistent truncation is still missing. Related to this, other points are not clear, as a classification of the sufficient conditions for a manifold to be Leibnitz parallelisable, or which are the 3-dimensional manifolds that do not admit a Leibnitz parallelisation.
Less related to the parallelisability concept, but still very important there are other obscure points, like how to include quantum aspects in generalised geometry, which is the precise relation between supersymmetry and generalised geometry, or further, how generalised geometry could help us to understand dualities and other still misterious aspects of string theory.
These are just some examples of the aims for an eventual prosecution of the study of generalised geometry, and already they seem to touch some very deep aspects of both mathematics and physics.

To sum up, we have seen how generalised geometry can be analysed to understand better supergravity, and how it seems to suggest also a way to deeper explore string theory. It is also a recent construction in pure mathematics, related to various areas beyond differential geometry, like algebraic topology, algebraic geometry and group theory. Thus it provides an example of a topic that lies at the frontier between mathematics and physics, on the one hand, receiving deep insights from both fields, but on the other hand it could also give some useful tools to understand and answer questions in both areas.
For these reasons, generalised geometry seems to be worth of further efforts and studies, since it may reveal something hidden so far and perhaps help us to understand the geometrical nature of string theory.

Appendix A

A detailed calculation for the Leibnitz parallelism of $S^2 \times S^1$

In this appendix we explain the details of the calculation for the algebra given in the example of a Leibnitz parallelisation of the product of spheres $S^2 \times S^1$ in section 3.3.2. On the one hand, we will see that the calculation is much simplified because the coordinates of the different factors do not mix. On the other hand, the fact that we are using constrained coordinates in the S^2 factor introduces some complications.

First, we construct the frames with some coefficients a, b, c to determine in order to find the right normalisation,

$$
\begin{aligned}
\hat{E}_i &= v_i - i_{v_i} B + a\, y_i \mathrm{d}\psi \; ; \\
\hat{E}'_i &= b(y_i \partial_\psi - i_{y_i \partial_\psi} B) + c\, \epsilon_{ijk} y^j \mathrm{d}y^k \; ,
\end{aligned}
$$

and then we impose the conditions to make them an $O(d, d)$ frame

$$
\begin{aligned}
\langle \hat{E}_i, \hat{E}_j \rangle &= \frac{1}{2}[i_{v_i}(a\, y_j \mathrm{d}\psi) + i_{v_j}(a\, y_i \mathrm{d}\psi)] = 0 \\
\langle \hat{E}'_i, \hat{E}'_j \rangle &= \frac{1}{2}[i_{b\, y_i \partial_\psi}(c\, \epsilon_{jkl} y^k \mathrm{d}y^l) + i_{b\, y_j \partial_\psi}(c\, \epsilon_{ikl} y^k \mathrm{d}y^l)] = 0 \\
\langle \hat{E}_i, \hat{E}'_j \rangle &= \frac{1}{2}[i_{v_i}(c\, \epsilon_{jkl} y^k \mathrm{d}y^l) + i_{b\, y_j \partial_\psi}(a\, y_i \mathrm{d}\psi)] \\
&= \frac{1}{2}(c\, \epsilon_{jkl} y^k i_{v_i} \mathrm{d}y^l + ab\, y_j y_i i_{\partial_\psi} \mathrm{d}\psi) = \\
&= \frac{1}{2}(c\, \epsilon_{ljk} y^k \epsilon_{lis} y^s + ab\, y_j y_i) = \\
&= \frac{1}{2}(c\, (\delta_{ji}\delta_{ks} - \delta_{js}\delta_{ki}) y^k y^s + ab\, y_j y_i) = \\
&= \frac{1}{2}(c\, \delta_{ij} y^s y^s - c\, y_i y_j + ab\, y_j y_i) = \frac{1}{2} c\, \delta_{ji} \; ,
\end{aligned}
$$

where in the last step we used that $y^i y^i = 1$.
Hence, imposing the condition on coefficients $ab = c$ we find that this frame is indeed an $O(d, d)$ frame,

$$\eta\left(\hat{E}_i, \hat{E}_j\right) = 0 \ , \qquad \eta\left(\hat{E}'_i, \hat{E}'_j\right) = 0 \ , \qquad \eta\left(\hat{E}_i, \hat{E}'_j\right) = \frac{1}{2}\delta_{ij} \ . \tag{A.1}$$

Next, we define our flux form as proportional to the wedge product of the volume forms of $\mathbb{R}^2$ and $\mathbb{R}^3$ projected onto the corresponding spheres,

$$H = h' \,\mathrm{vol}_2 \wedge \mathrm{vol}_1 \ .$$

To construct it first we need to calculate these projections as follows,

$$\begin{aligned}
\mathrm{vol}_1 &= i_{\partial_r} \mathrm{vol}_{\mathbb{R}^2}\big|_{r=R_1} = \frac{y^k}{r} i_{\partial_k}\left(\frac{r^2}{2!}\epsilon_{ij}\mathrm{d}y^i \wedge \mathrm{d}y^j\right)\big|_{r=R_1} = \\
&= \frac{r}{2}\epsilon_{ij}(y^i \mathrm{d}y^j - y^j \mathrm{d}y^i)\big|_{r=R_1} = r\epsilon_{ij} y^i \mathrm{d}y^j\big|_{r=R_1} = R_1 \mathrm{d}\psi \ ,
\end{aligned}$$

$$\begin{aligned}
\mathrm{vol}_2 &= i_{\partial_r} \mathrm{vol}_{\mathbb{R}^3}\big|_{r=R_2} = \frac{y^l}{r} i_{\partial_l}\left(\frac{r^3}{3!}\epsilon_{ijk}\mathrm{d}y^i \wedge \mathrm{d}y^j \wedge \mathrm{d}y^k\right)\big|_{r=R_2} = \\
&= \frac{r^2}{3!}\epsilon_{ijk}(y^i \mathrm{d}y^j \wedge \mathrm{d}y^k - y^j \mathrm{d}y^i \wedge \mathrm{d}y^k + y^k \mathrm{d}y^i \wedge \mathrm{d}y^j)\big|_{r=R_2} = \\
&= \frac{3r^2}{3!}\epsilon_{ijk} y^i \mathrm{d}y^j \wedge \mathrm{d}y^k\big|_{r=R_2} = \frac{R_2^2}{2}\epsilon_{ijk} y^i \mathrm{d}y^j \wedge \mathrm{d}y^k \ .
\end{aligned}$$

Putting all together, and redefining the proportionality constant, we find

$$H = h' \frac{R_1 R_2^2}{2}\epsilon_{ijk} y^i \, dy^j \wedge dy^k \wedge d\psi = \frac{h}{2}\epsilon_{ijk} y^i \, dy^j \wedge dy^k \wedge d\psi \ . \tag{A.2}$$

In order to compute the Dorfman Derivatives we need to calculate how each part of the frames transforms under the action of all the others. We should so calculate the expressions fo Lie derivatives. All these calculations are detailed in the following

$$\begin{aligned}
&[v_i, v_j] = -\epsilon_{ijk} v_k \ ; \\
&i_{v_i} dy_j = dy_j(\epsilon_{ikl} y_k \partial_l) = \epsilon_{ikl} y_k \delta_{jl} = -\epsilon_{ijk} y_k \ ; \\
&\mathcal{L}_{v_i} y_j = i_{v_i} \mathrm{d}y_j = -\epsilon_{ijk} y_k \ ; \\
&\mathcal{L}_{v_i} dy_j = \mathrm{d}(i_{v_i} \mathrm{d}y_j) = \mathrm{d}(-\epsilon_{ijl} y_l) = -\epsilon_{ijl} \mathrm{d}y_l \ ; \\
&\mathcal{L}_{v_i} y_j \partial_\psi = [v_i, y_j \partial_\psi] = (\mathcal{L}_{v_i} y_j)\partial_\psi = -\epsilon_{ijk} y_k \partial_\psi \ ; \\
&\mathcal{L}_{v_i} \epsilon_{jkl} y^k \mathrm{d}y^l = -\epsilon_{ijk}\epsilon_{krl} y^r \mathrm{d}y^l \ .
\end{aligned}$$

Where the last is less immediate and follows from

$$\mathcal{L}_{v_i}\epsilon_{jkl}y^k \mathrm{d}y^l = \epsilon_{jkl}[(\mathcal{L}_{v_i}y^k)\mathrm{d}y^l + (\mathcal{L}_{v_i}\mathrm{d}y^l)y^k] = \epsilon_{jkl}(-\epsilon_{ikr}y^r\mathrm{d}y^l - \epsilon_{ilr}y^k\mathrm{d}y^r) =$$
$$= -(\epsilon_{jkl}\epsilon_{ikr} + \epsilon_{jrk}\epsilon_{ikl})y^r\mathrm{d}y^l = -\epsilon_{ijk}\epsilon_{krl}y^r\mathrm{d}y^l \ .$$

Note that in the last calculation we used that the ϵ symbols are the structure constants of $\mathfrak{so}(3)$, and from the Jacobi identity they satisfy

$$\epsilon_{jlk}\epsilon_{kir} + \epsilon_{kjr}\epsilon_{lik} + \epsilon_{ijk}\epsilon_{klr} = 0 \ .$$

Also, we need to calculate the inner products of the vector parts with the flux form, since we will be using the Dorfman derivative coming from the twisted Courant bracket, in the equation (1.24)

$$\begin{aligned} i_{by_i\partial_\psi}H &= \frac{bh}{2}y_i\epsilon_{lmn}y^l i_{\partial_\psi}(\mathrm{d}y^n\wedge\mathrm{d}y^m\wedge\mathrm{d}\psi) = \\ &= \frac{bh}{2}y_i\epsilon_{lmn}y^l\mathrm{d}y^n\wedge\mathrm{d}y^m = \\ &= \frac{bh}{2}\epsilon_{imn}y_l\, y^l\mathrm{d}y^n\wedge\mathrm{d}y^m = \\ &= \frac{bh}{2}\epsilon_{lmn}\mathrm{d}y^n\wedge\mathrm{d}y^m \ , \end{aligned}$$

where we used the following,

$$y_{[i}\epsilon_{lmn]} = \frac{1}{4!}(y_i\epsilon_{lmn} - y_l\epsilon_{mni} + y_m\epsilon_{nli} - y_n\epsilon_{lm}) = 0 \ ,$$

and again $y^iy^i = 1$.

For the v_i contraction of H we have

$$\begin{aligned} i_{v_i}H &= \frac{h}{2}\epsilon_{lnm}y^l i_{v_i}(\mathrm{d}y^n\wedge\mathrm{d}y^m\wedge\mathrm{d}\psi) = \\ &= \frac{h}{2}\epsilon_{lnm}y^l(\mathrm{d}y^n(v_i)\mathrm{d}y^m - \mathrm{d}y^m(v_i)\mathrm{d}y^n)\wedge\mathrm{d}\psi = \\ &= \frac{h}{2}\epsilon_{lnm}y^l(-\epsilon_{inj}y^j\mathrm{d}y^m + \epsilon_{imj}y^j\mathrm{d}y^n)\wedge\mathrm{d}\psi = \\ &= \frac{h}{2}\epsilon_{lmn}y^l(2\epsilon_{inj}y^j\mathrm{d}y^m)\wedge\mathrm{d}\psi = \\ &= h\,\epsilon_{nlm}\epsilon_{nji}y^ly^j\mathrm{d}y^m\wedge\mathrm{d}\psi = \\ &= h(\delta_{lj}\delta_{mi} - \delta_{li}\delta_{mj})y^ly^j\mathrm{d}y^m\wedge\mathrm{d}\psi = \\ &= h(y^jy^j dy^i - y^iy^j dy^j)\wedge\mathrm{d}\psi = \\ &= h\,\mathrm{d}y^i\wedge\mathrm{d}\psi \ . \end{aligned}$$

Exploiting again the fact that $y^i y^i = 1$, which implies $\mathrm{d}y^i y^i = 0$.

Finally, we are ready to calculate the algebra relations using all these results,

$$
\begin{aligned}
L_{\hat{E}_i}\hat{E}_j &= [v_i, v_j] - i_{[v_i,v_j]} + \mathcal{L}_{v_i} a y_j \mathrm{d}\psi + i_{v_j}(i_{v_i}H + \mathrm{d}(a y_i \mathrm{d}\psi)) = \\
&= -\epsilon_{ijk}v_k - i_{-\epsilon_{ijk}v_k}B + a(\mathcal{L}_{v_i}y_j)\mathrm{d}\psi + i_{v_j}(h\mathrm{d}y^i \wedge \mathrm{d}\psi + a\mathrm{d}y_i \wedge \mathrm{d}\psi) = \\
&= -\epsilon_{ijk}(v_k - i_{v_k}B + a\epsilon_{ijk}y_k\mathrm{d}\psi) + (a+h)i_{v_j}\mathrm{d}y_i \wedge \mathrm{d}\psi = \\
&= -\epsilon_{ijk}\hat{E}_k + (a+h)\epsilon_{ijk}y_k\mathrm{d}\psi = \\
&= -\epsilon_{ijk}\hat{E}_k \ ,
\end{aligned}
$$

$$
\begin{aligned}
L_{\hat{E}'_i}\hat{E}'_j &= [b\, y_i\partial_\psi, b\, y_j\partial_\psi] - i_{[by_i\partial_\psi, by_j\partial_\psi]}B + \\
&+ \mathcal{L}_{by_i\partial_\psi}(c\,\epsilon_{jkl}y^k\mathrm{d}y^l) + i_{by_j\partial_\psi}(i_{by_i\partial_\psi}H + \mathrm{d}(c\,\epsilon_{ikl}y^k\mathrm{d}y^l)) = \\
&= 0 - i_0B + b\,c\,\epsilon_{jkl}\mathcal{L}_{y_j\partial_\psi}(y^k\mathrm{d}y^l) + i_{b\,y_j\partial_\psi}(\frac{bh}{2}\epsilon_{ikl}\mathrm{d}y^k \wedge \mathrm{d}y^l + c\,\epsilon_{ikl}\mathrm{d}y^k \wedge \mathrm{d}y^l) = \\
&= (\frac{bh}{2} + c)\epsilon_{ikl}b\,y_j i_{\partial_\psi}(\mathrm{d}y^k \wedge \mathrm{d}y^l) = 0 \ ,
\end{aligned}
$$

$$
\begin{aligned}
L_{\hat{E}_i}\hat{E}'_j &= [v_i, by_j\partial_\psi] - i_{[v_i,by_j\partial_\psi]}B + \mathcal{L}_{v_i}(c\,\epsilon_{jlm}y^l\mathrm{d}y^m) + i_{by_j\partial_\psi}(i_{v_i}H + \mathrm{d}(ay_i\mathrm{d}\psi)) = \\
&= -b\,\epsilon_{ijk}y_k\partial_\psi - i_{-b\epsilon_{ijk}y_k\partial_\psi}B - c\,\epsilon_{ijk}\epsilon_{klm}(y^l\mathrm{d}y^m) + (a+h)i_{by_j\partial_\psi}\mathrm{d}y_i \wedge \mathrm{d}\psi = \\
&= -\epsilon_{ijk}\hat{E}'_k - b(a+h)y_j\mathrm{d}y_i = -\epsilon_{ijk}\hat{E}'_k \ .
\end{aligned}
$$

So imposing the condition $a = -h$ we find the $\mathfrak{iso}(3)$ algebra relations, related to the transformations of $SO(3) \ltimes \mathbb{R}^3$, the euclidean isometry group,

$$
\begin{aligned}
L_{\hat{E}_i}\hat{E}_j &= [\![\hat{E}_i, \hat{E}_j]\!] = -\epsilon_{ijk}\hat{E}_k \ ; && \text{(A.3)} \\
L_{\hat{E}_i}\hat{E}'_j &= [\![\hat{E}_i, \hat{E}'_j]\!] = -\epsilon_{ijk}\hat{E}'_k \ ; && \text{(A.4)} \\
L_{\hat{E}'_i}\hat{E}'_j &= [\![\hat{E}'_i, \hat{E}'_j]\!] = 0 \ . && \text{(A.5)}
\end{aligned}
$$

As a last step, we fix the remaining normalisation constants as $a = b = c = -h = 1$, such that the frame assumes the usual form

$$
\begin{aligned}
\hat{E}_i &= v_i - i_{v_i}B + \ y_i\mathrm{d}\psi \\
\hat{E}'_i &= y_i\partial_\psi - i_{y_i\partial_\psi}B + \epsilon_{ijk}y^j\mathrm{d}y^k
\end{aligned}
$$

and for the flux three-form we recover the (3.24),

$$
H = -\frac{1}{2}\epsilon_{ijk}y^i\mathrm{d}y^j \wedge \mathrm{d}y^k \wedge \mathrm{d}\psi \ .
$$

In this way we obtained a globally defined generalised frame, defining a general Leibnitz parallelisation of $S^2 \times S^1$. Furthermore, we discovered that the algebra related to this parallelism is the $\mathfrak{iso}(3)$, encoding the symmetries of the 3-dimensional euclidean space.

To conclude this appendix we just note that we used the δ_{ij} to raise and lower indeces on the coordinates y_i, actually, the euclidean metric should be $1/R^2\delta_{ij}$ since it should come from the coordinate transformation $x^i = y^i/R$, but since the two tensor differs just modulo a constant parameter R, we should ignore it for our purposes.

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